Topologie

X mulțime; $\tau\subseteq\wp(X)$

(T1) $\emptyset\in\tau$; $X\in\tau$

(T2) $\forall(M_j)\subseteq\tau \Rightarrow \cup M_j\in\tau$

(T3) $\forall(M_j)\subseteq\tau$, finită $\Rightarrow \cap M_j\in\tau$

M deschisă $\Leftrightarrow M\in\tau$

Topologia discretă și indiscretă

$\tau_{dis}:=\wp(X)$; $\tau_{indis}:=\{\emptyset,X\}$

Topologia uzuală pe $K\in\{\Re,C\}$

$B(x_0,r_0)=\{x\in K \mid |x-x_0|<r_0\}$

$\tau_u=\{M\subseteq K \mid \forall x\in M\ \exists r>0 : B(x,r)\subseteq M\}$

Mulțimi închise

(X,τ) sp.t.; $M\subseteq X$ închisă$\Leftrightarrow C_M X\in\tau$; $\underline{\tau} = \{M\in X \mid C_M X\in\tau\}$; $\underline{\tau}\subseteq\wp(X)$

P.1.1 Proprietățile $\underline{\tau}$

($\underline{\tau}$1) $\emptyset\in\underline{\tau}$; $X\in\underline{\tau}$

($\underline{\tau}$2) $\forall(M_j)\subseteq\underline{\tau} \Rightarrow \cap M_j\in\underline{\tau}$

($\underline{\tau}$3) $\forall(M_j)\subseteq\tau$, finită $\Rightarrow \cup M_j\in\underline{\tau}$

Vecinătate

(X,τ) sp.t.; $\mathcal{V}(x)=\{V\subseteq X \mid \exists M\in\tau : x\in M\subseteq V\}$

$V\in\mathcal{V}(x)$ vecinătate a lui x

Interiorul, închiderea și densitatea unei mulțimi

$IntA=\mathring{A}=\{x\in A \mid A\in\mathcal{V}(x)\}$; $clA=\tilde{A}=\{x\in X \mid \forall V\in\mathcal{V}(x)\Rightarrow V\cap A\neq\emptyset\}$

$A\subseteq B\subseteq X$ densă în B $\Leftrightarrow B\subseteq clA$; A densă în X $\Leftrightarrow clA=X$;

(?) $\underline{\tau}=\{clA,\ A\subseteq X\}\cup\{\emptyset\}$

Spații separate Hausdorff

(X,τ) s.s.H $\Leftrightarrow (X,\tau)$ sp.t.; $\forall x\neq y\in X, \exists V\in\mathcal{V}(x), U\in\mathcal{V}(y) : V\cap U=\emptyset$;

T.1.2 Proprietățile $\mathcal{V}(x)$

(V1) $\mathcal{V}(x)\neq\emptyset$; (V2)$\forall V\in\mathcal{V}(x), x\in V$

(V3) $\forall V\in\mathcal{V}(x), \forall W\subseteq X, V\subseteq W \Rightarrow W\in\mathcal{V}(x)$

(V4) $\forall(V_j)\subseteq\mathcal{V}(x)$ finită $\Rightarrow \cap V_j\in\mathcal{V}(x)$

(V5) $\forall V\in\mathcal{V}(x), \exists W\subseteq \mathcal{V}(x), \forall y\in W \Rightarrow V\in\mathcal{V}(y)$

P.1.3 Topologie vs. Vecinătăți

(X,τ) sp.t.; $M\subseteq X$; $M\in\tau \Leftrightarrow \forall x\in M, M\in\mathcal{V}(x)$;

Sistem fundamental de vecinătăți $\mathcal{U}$(x) s.f.v. x

(X,τ) sp.t.; $\mathcal{U}(x)\subseteq\wp(X)$ s.f.v.(bază) a lui x $\Leftrightarrow \mathcal{U}(x)\subseteq\tau$; $\mathcal{U}(x)\subseteq\mathcal{V}(x)$;

$(\mathcal{U}(x)\subseteq\mathcal{V}(x) \Leftrightarrow \forall V\in\mathcal{V}(x)\ \exists U\in\mathcal{U}(x) : U\subseteq V)$;

P.1.4 $\mathcal{U}$(x) vs. $\mathcal{V}$(x)

$\mathcal{U}(x)$ s.f.v. x $\Leftrightarrow \mathcal{V}(x)=\{V\subseteq X \mid \exists U\in\mathcal{U}(x), U\subseteq V\}$

T.1.5 Construcția τ prin funcții de vecinătăți

$X\neq\emptyset$; $F:X\to\wp(\wp(X))$ cu $\forall x\in X$:

(F1) $F(x)\neq\emptyset$; (F2) $\forall V\in F(x) \Rightarrow x\in V$

(F3) $\forall V\subseteq F(x), \forall W\subseteq\wp(X), V\subseteq W \Rightarrow W\in F(x)$

(F4) $\forall(V_j)\subseteq F(x)$ finită $\Rightarrow \cap V_j\in F(x)$

(F5) $\forall V\in F(x)\ \exists W\in F(x) \mid \forall y\in W \Rightarrow V\in F(y)$

$\Rightarrow \exists!\ \tau_F$ pe X a.î. $\forall x\ \mathcal{V}(x)=F(x)$; F - funcție de vecinătăți;

$\tau_F:=\{M\subseteq X \mid \forall x\in M \Rightarrow M\in F(x)\}$; τ_F top. indusă de F pe X;

T.1.6 Construcția τ funcții de bază (s.f.v.)

$X\neq\emptyset$; $\mathcal{U}:X\to\wp(\wp(X))$ cu $\forall x\in X$:

(U1) $\mathcal{U}(x)\neq\emptyset$; (U2) $\forall U\in\mathcal{U}(x) \Rightarrow x\in U$

(U3) $\forall U_1,U_2\subseteq\mathcal{U}(x), \exists U\in\mathcal{U}(X) : U\subseteq U_1\cap U_2$

(U4) $\forall U\in\mathcal{U}(x)\ \exists U'\in\mathcal{U}(x) \mid \forall y\in U'\ \exists U''\in\mathcal{U}(y) : U''\subseteq U$

$\Rightarrow \exists!\ \tau$ pe X a.î. $\forall x\in X$, $\mathcal{U}(x)$ s.f.v. x; $\mathcal{U}$ - funcție de baze;

$F(x):=\{V\subseteq X \mid \exists U\in\mathcal{U}(x) : U\subseteq V\}$; F funcția de vecinătăți indusă de $\mathcal{U}$

$\tau_{\mathcal{U}}:=\{M\subseteq X \mid \forall x\in M, \exists U\in\mathcal{U}(x): U\subseteq M\}$; $\tau_{\mathcal{U}}$ top. indusă de $\mathcal{U}$ pe X;

Compactitate

Fie (X,τ_X) sp. t.; $\mathcal{A}=(A_i)_{i\in I}$;

$(Y,\tau_X|_Y)\leq(X,\tau_X)$; $\mathcal{A}$.*ac*.Y $\Leftrightarrow Y\subseteq\cup_{i\in I}A_i$; ac.=acoperire

A.*ac*.X $\Leftrightarrow X=\cup_{i\in I}A_i$; ac.=acoperire

I finită$\Leftrightarrow\mathcal{A}$ ac. finită; $A_i\in\tau \Leftrightarrow \mathcal{A}$ ac. deschisă; $A_i\in\underline{\tau} \Leftrightarrow \mathcal{A}$ ac. închisă;

$Y\subseteq X$ compact$\Leftrightarrow$

$$\forall\mathcal{A}=(A_i)_{i\in I}.\text{ac}.Y, \mathcal{A}\subseteq\tau, \exists B=(B_j)_{j\in J}.\text{ac}.Y, |J|<\aleph_0 \text{ a.î. } \forall i, \exists j, A_i=B_j;$$

T.1.7 (X,τ_X) sp.t.; $Y\subseteq X$ compactă $\Leftrightarrow (Y,\tau_X|_Y)$ compact;

T.1.8 (X,τ_X) sp.t.; $Y\in\underline{\tau}_X \Rightarrow$ Y compactă;

T.1.9 (X,τ_X) s.s.H.; Y compactă $\Rightarrow Y\in\underline{\tau}_X$;

Obs. Un spațiu discret este compact numai dacă este finit;

Spațiu liniar pe $K\in\{\Re,C\}$

$(X,K,+,\cdot)$ s.l. peste K, unde

$X\neq\emptyset$; $\forall x,y\in X,\forall\alpha\in K\Rightarrow(x+y)\in X;\alpha\cdot x\in X$

cu

(SL1) $\forall x,y,z\in X$ $(x+y)+z=x+(y+z)$ (as+)

(SL2) $\exists O_X\in X : \forall x\in X \Rightarrow x+O_X=x$ (O_X)

(SL3) $\forall x\in X$ $\exists -x\in X : x+(-x)=O_X$ $(-x)$

(SL4) $\forall x,y\in X : x+y=y+x$ (com+)

(SL5) $\forall x\in X$ $1\cdot x=x$ $(1_K=1_X)$

(SL6) $\forall\alpha,\beta\in K,\forall x\in X$ $(\alpha\beta)\cdot x=\alpha\cdot(\beta\cdot x)$ (as·)

(SL7) $\forall\alpha,\beta\in K,\forall x\in X$ $(\alpha+\beta)\cdot x=\alpha\cdot x+\beta\cdot x$ (XdisK)

(SL8) $\forall\alpha\in K,\forall x,y\in X$ $\alpha\cdot(x+y)=\alpha\cdot x+\alpha\cdot y$ (KdisX)

atunci $(X,K,+,\cdot)$ s.l.;

T.2.1 O_X şi –x în s.l.

$(X,K,+,\cdot)$ s.l. $\Rightarrow$

(1) $\exists! O_X$; $\forall x\in X$ $0\cdot x=O_X$; $\forall\alpha\in K$ $\alpha\cdot O_X=O_X$;

(2) $\exists!$ $(-x)$; $(-x)=(-1)\cdot x$;

Ex.*s.l.*1 $X=K^m=K\times K\times\ldots\times K$; $(K^m,K,+,\cdot)$ s.l.

$x=(x_1,\ldots,x_m)$; $y=(y_1,\ldots,y_m)$;

$x+y=(x_1+y_1,\ldots,x_m+y_m)$; $\alpha\cdot x=(\alpha\cdot x_1,\ldots,\alpha\cdot x_m)$;

$(K^m,K,+,\cdot)$ s.l.; $O_K{}^m=(0,\ldots,0)$; $(-x)=(-x_1,\ldots,-x_m)$;

Ex.*s.l.*2 $X=(K)=(K_j)_{j\geq1}$ şiruri; $((K),K,+,\cdot)$ s.l.

$x=(x_i)_{i\geq1}$; $y=(y_i)_{i\geq1}$; $x+y=(x_i+y_i)_{i\geq1}$; $\alpha\cdot x=(\alpha x_i)_{i\geq1}$;

$O_{(K)}=(0)_{i\geq1}$; $(-x)=(-x_i)_{i\geq1}$

Ex.*s.l.*3 $(U,K,+,\cdot)$ s.l. $\Rightarrow$ F(T,U) s.l.; F(T,K) s.l.

$(U,K,+,\cdot)$ s.l.; $T\neq\emptyset$; $F(T,U):=\{F \mid F:T\to U\}$ funcții;

$x,y:T\to U$; $\alpha\in K$; $(x+y):T\to U$, $(x+y)(t):=x(t)+y(t)$;

$(\alpha\cdot x):T\to U$, $(\alpha\cdot x)(t):=\alpha\cdot x(t)$;

$O_{T\to U}:T\to U$, $(O_{T\to U})(t):=O_U$; $(-x):T\to U$ $(-x)(t):=-x(t)$;

$\Rightarrow (F(T,U),K,+,\cdot)$ s.l.;

$U=K$; $T=\{1,\ldots,m\} \Rightarrow (K^m,K,+,\cdot)$ s.l.; $T=(i)_{i\geq1} \Rightarrow ((K),K,+,\cdot)$ s.l.;

Operații cu mulțimi în s.l.

$(X,K,+,\cdot)$ s.l.; $M_i \in \wp(X)$; $\alpha_i \in K$;

$\alpha_1 M_1+\ldots+\alpha_m M_m=\{x\in X \mid \exists x_i\in M_i : x=\Sigma\alpha_i\cdot x_i\}$

$M_1+M_2=\{x\in X \mid \exists x_1\in M_1, \exists x_2\in M_2 : x=x_1+x_2\}$

$x_1+M_2=\{x_1\}+M_2=\{x\in X \mid \exists x_2\in M_2 : x=x_1+x_2\}$

$\alpha_1 M_1=\{x\in X \mid \exists x_1\in M_1 : x=\alpha_1\cdot x_1\}$

$-M_1=(-1)M_1=\{x\in X \mid \exists x_1\in M_1 : x = -x_1\}$

$M_1-M_2=\{x\in X \mid \exists x_1\in M_1, \exists x_2\in M_2 : x=x_1-x_2\}$

$M-M=\{x\in X \mid \exists x_1,x_2\in M : x=x_1-x_2\}$

Subspațiu liniar, M≤X:

$(M,K,+,\cdot) \le (X,K,+,\cdot) \Leftrightarrow M\neq\emptyset;\ \forall\alpha_1,\alpha_2\in K\ \alpha_1 M+\alpha_2 M\subseteq M$

T.2.2 Caracterizarea s.s.l.

$(M,K,+,\cdot) \le (X,K,+,\cdot) \Leftrightarrow$

(SSL1) $M\neq\emptyset$

(SSL2) $M+M\subseteq M$

(SSL3) $\forall\alpha\in K \Rightarrow \alpha M\subseteq M$

P.2.3 Generalizarea caracterizării s.s.l.

$(M,K,+,\cdot) \le (X,K,+,\cdot) \Leftrightarrow M\neq\emptyset;\ \forall\alpha_i\in K \Rightarrow \Sigma\alpha_i M\subseteq M$

Ex.s.s.l.

$p>0$; $l_p=\{(x_i)_{i\ge1}\in(K) : \Sigma|x_i|^p<\infty\}$; $(l_p,K,+,\cdot)\le((K),K,+,\cdot)$

$(0)\in l_p\neq\emptyset$; $\forall\alpha_1,\alpha_2\in K$, $\forall(x_i)_{i\ge1},(y_i)_{i\ge1}\in l_p$ avem:

$\alpha_1\cdot(x_i)_{i\ge1}+\alpha_2\cdot(y_i)_{i\ge1}=(\alpha_1 x_i+\alpha_2 y_i)_{i\ge1}$

$|\alpha_1 x_i+\alpha_2 y_i|\le|\alpha_1 x_i|+|\alpha_2 y_i|=|\alpha_1|\cdot|x_i|+|\alpha_2|\cdot|y_i|\le$

$\le|\alpha_1|\cdot\max(|x_i|,|y_i|)+|\alpha_2|\cdot\max(|x_i|,|y_i|)$

$|\alpha_1 x_i+\alpha_2 y_i|^p\le(|\alpha_1|+|\alpha_2|)^p\cdot\max(|x_i|^p,|y_i|^p)\le$

$\le(|\alpha_1|+|\alpha_2|)^p(|x_i|^p+|y_i|^p)=(|\alpha_1|+|\alpha_2|)^p|x_i|^p+(|\alpha_1|+|\alpha_2|)^p|y_i|^p$

$\Sigma|\alpha_1 x_i+\alpha_2 y_i|^p\le(|\alpha_1|+|\alpha_2|)^p\Sigma|x_i|^p+(|\alpha_1|+|\alpha_2|)^p\Sigma|y_i|^p$;

$\alpha_1,\alpha_2<\infty$; $\Sigma|x_i|^p,\Sigma|y_i|^p<\infty \Rightarrow \Sigma|\alpha_1 x_i+\alpha_2 y_i|^p<\infty$;

Obs. s.s.l.

$(X,K,+,\cdot)$ s.l.; $(M,K,+,\cdot) \le (X,K,+,\cdot) \Rightarrow (M,K,+,\cdot)$ s.l.;

Învelitoare liniară

$(X,K,+,\cdot)$ s.l.; $M\subseteq X$;

$\mathrm{lin}M:=\cap\{S\subseteq X : (S,K,+,\cdot)\le(X,K,+,\cdot), M\subseteq S\}$;

$lin\{x\}=\cap\{S\subseteq X : (S,K,+,\cdot)\leq(X,K,+,\cdot), x\in S\}$;

P.2.4 Caracterizarea învelitorii liniare

(IL1) $M\subseteq linM$; (IL2) $(linM,K,+,\cdot)\leq(X,K,+,\cdot)$;

(IL3) $(S,K,+,\cdot)\leq(X,K,+,\cdot)$, $M\subseteq S \Rightarrow linM\subseteq S(\leq)$;

Obs.IL: $M=linM \Leftrightarrow (M,K,+,\cdot) \leq (X,K,+,\cdot)$

T.2.5 Construcția (reprezentarea) învelitorii liniare

$(X,K,+,\cdot)$ s.l.; $M\subseteq X$;

$linM=\{x\in X \mid \exists\alpha_i\in K,\exists x_i\in M : x=\Sigma\alpha_i\cdot x_i\}$; x cmb.lin. M;

Ex. $\wp(\wp(F(T,K)))$ topologia convergenței uniforme

$F(T,K)=\{f \mid f:T\to K\}$; $B_T(f_0,r_0)=\{f\in F(T,K) : \forall t\in T, |f(t)-f_0(t)|<r_0\}$;

$U:F(T,K)\to\wp(\wp(F(T,K)))$; $U(f)=\{B_T(f,r), 0<r<\infty\}$;

$\Rightarrow U$ bază indusă pe F(T,K) (?):

$O_F:T\to K$, $O_F(t)=0$;

$\forall t\in T\ |O_F(t)-O_F(t)|=0 \Rightarrow \exists B_T(O_F,r=1)\neq\emptyset \Rightarrow U(f)\neq\emptyset$;

$\forall U=B_T(f,r)$, $|f(t)-f(t)|=0 \Rightarrow f\in B_T(f,r)=U$;

$\forall U_1=B_1(f,r_1),U_2=B_2(f,r_2)\in U(f)$; $U:=B_T(f,\min(r_1,r_2))$

$\min(r_1,r_2)\leq r_1,r_2 \Rightarrow U\subseteq U_1,U_2 \Rightarrow U\subseteq U_1\cap U_2$

$U=B_T(f,r)$; $U':=B_T(f,r/2)$; $\forall y\in U' \Rightarrow |y(t)-f(t)|<r/2$

$U(y)=\{B_T(y,r),0<r<\infty\}$; $U''=B_T(y,r/2)$

(?) $B_T(y,r/2)\subseteq B_T(f,r)$; $z\in B_T(y,r/2) \Rightarrow |z(t)-y(t)|<r/2$

$|z(t)-f(t)|=|z(t)-y(t)+y(t)-f(t)|\leq|z(t)-y(t)|+|y(t)-f(t)|$

(U4) $\forall U\in U(f)\ \exists U'\in U(f) \mid \forall y\in U'\ \exists U''\in U(y) : U''\subseteq U$

$B_T(f,r)=U\in U(f)$; $y\in U':=B_T(f,r/2) \Rightarrow |y(t)-f(t)|<r/2$;

$z\in U'':=B_T(y,r/2) \Rightarrow |z(t)-y(t)|<r/2$;

$|z(t)-f(t)|=|z(t)-y(t)+y(t)-f(t)|\leq |z(t)-y(t)|+|y(t)-f(t)|<r \Rightarrow$

$z\in B_T(f,r)=U \Rightarrow U''\subseteq U$

$\Rightarrow U$ bază pe F(T,K) $\Rightarrow \exists!\ \tau_U$ topologie pe F(T,K) a.î. în $(F(T,K),\tau_U)$,

$\forall x\in F(T,K) \Rightarrow U(x)$ s.f.v. (bază) a lui x

T.2.6 τ_U generată de $B_T(x,r)$ pe F(T,K) topologia conv. uniforme

$(f_i) \subseteq F(T,K) \to f \in F(T,K) \Leftrightarrow f_i$ conv. absolut la f $\Leftrightarrow$

$\forall\varepsilon, \exists i_0 : \forall i\geq i_0, \forall t\in T : |f_i(t)-f_{i0}(t)|<\varepsilon$

Semimetrică

$X\neq\emptyset$; $sm:X\times X\to\Re$ semimetrică pe X $\Leftrightarrow$

(SM1) $\forall x\in X \Rightarrow sm(x,x)=0$

(SM2) $\forall x,y\in X \Rightarrow sm(x,y)=sm(y,x)$

(SM3) $\forall x,y,z\in X \Rightarrow sm(x,y)\leq sm(x,z)+sm(z,y)$

Obs.SM $0=d(x,x)\leq d(x,y)+d(y,x)=2d(x,y) \Rightarrow d(x,y)\geq 0$

Metrică

$X\neq\emptyset$; $m:X\times X\rightarrow\Re$ metrică pe X $\Leftrightarrow$

m semimetrică pe X şi ($x\neq y \Rightarrow m(x,y)>0$)

Bile determinate de semimetrică

$B_{sm}(x_0,r_0)=\{x\in X : sm(x,x_0)<r_0\}$

$\underline{B}_{sm}(x_0,r_0)=\{x\in X : sm(x,x_0)\leq r_0\}$

T.3.1 Topologia generată de o semimetrică

$X\neq\emptyset$; $sm:X\times X\rightarrow\Re$ semimetrică pe X;

$\tau_{sm}:=\{M\subseteq X \mid \forall x\in M\ \exists r>0 : B_{sm}(x,r)\subseteq M\}$ top. pe X; (X,τ_{sm}) sp.sm.;

D. $\emptyset\in\tau_{sm}$; $X\in\tau_{sm} \Leftrightarrow \forall x\in X\ \exists r>0 : B_{sm}(x,r)\subseteq X \Leftrightarrow$

$\forall x\in X\ \exists r>0\ \{x\in X : sm(x,x_0)<r_0\}\subseteq X$ evident

(T2) $M_j\in\tau_{sm} \Rightarrow \forall x\in M_j\ \exists r_j>0 : B_{sm}(x,r_j)\subseteq M_j$

(?) $\cup M_j\in\tau_{sm} \Leftrightarrow \forall x\in\cup M_j\ \exists r>0 : B_{sm}(x,r)\subseteq\cup M_j$

$x\in\cup M_j \Rightarrow \exists i : x\in M_i\in\tau_{sm} \Rightarrow$

$\Rightarrow \exists r=r_i>0 : B_{sm}(x,r_i)\subseteq M_i\subseteq\cup M_j \Rightarrow \cup M_j\in\tau_{sm}$

(T3) $M_j\in\tau_{sm} \Rightarrow \forall x\in M_j\ \exists r_j>0 : B_{sm}(x,r_j)\subseteq M_j$

(?)$\cap M_j\subseteq\tau_{sm}$ finită$\Leftrightarrow \forall x\in\cap M_j\ \exists r>0 : B_{sm}(x,r)\subseteq\cap M_j$

$x\in\cap M_j \Rightarrow x\in M_j\in\tau_{sm} \Rightarrow \exists r_j>0 : B_{sm}(x,r_j)\subseteq M_j$;

M_j finită, $r:=\min(r_j) \Rightarrow B_{sm}(x,r)\subseteq M_j \Rightarrow B_{sm}(x,r)\subseteq\cap M_j$

P.3.2 Bile închise şi deschise în sp. sm.

$\tau_{sm}:=\{M\subseteq X \mid \forall x\in M\ \exists r>0 : B_{sm}(x,r)\subseteq M\}$

$B_{sm}(x_0,r_0)\in\tau_{sm} \Leftrightarrow \forall x\in B(x_0,r_0)\ \exists r>0 :$

$B_{sm}(x,r)\subseteq B_{sm}(x_0,r_0)$; $x\in B(x_0,r_0) \Rightarrow sm(x,x_0)<r_0$;

$r:=r_0-sm(x,x_0)>0$; $y\in B_{sm}(x,r) \Rightarrow sm(x,y)<r_0-sm(x,x_0) \Rightarrow$

$sm(x,y)+sm(x,x_0)<r_0 \Rightarrow sm(x_0,y)<r_0 \Rightarrow y\in B_{sm}(x_0,r_0)$

$\underline{B}_{sm}(x_0,r_0)\in\underline{\tau}_{sm} \Leftrightarrow C_X\underline{B}_{sm}(x_0,r_0)\in\tau_{sm} \Leftrightarrow$

$\forall x\in C_XB_{sm}(x_0,r_0)\ \exists r>0 : B_{sm}(x,r)\subseteq C_XB_{sm}(x_0,r_0)$

$x\in C_XB_{sm}(x_0,r_0) \Rightarrow sm(x,x_0)>r_0$; $r:=sm(x,x_0)-r_0>0$

$y\in B_{sm}(x,r) \Rightarrow sm(x,y)<sm(x,x_0)-r_0 \Leftrightarrow$

$r_0 < sm(x,x_0) - sm(x,y) < sm(x_0,y) \Rightarrow y \in C_X B_{sm}(x_0,r_0)$

P.3.3 Proprietățile bilelor *vs.* vecinătăți

(X,sm) sp. sm.; $x \in X$; $V \subseteq X$; u.a.s.e.:

$V \in \mathcal{V}(x) \Leftrightarrow \exists r>0 : B(x,r) \subseteq V \Leftrightarrow \exists r>0 : \underline{B}(x,r) \subseteq V$

$V \in \mathcal{V}(x) \Leftrightarrow \exists M \in \tau_{sm} : x \in M \subseteq V$;

$M \in \tau_{sm} \Rightarrow \forall y:=x \in M \; \exists r>0 : B_{sm}(y=x,r) \subseteq M \subseteq V$

$\exists r>0 : B_{sm}(x,r) \subseteq V$; $r'=r/2$; $y \in \underline{B}_{sm}(x,r/2) \Rightarrow$

$sm(x,y) \leq r/2 < r \Rightarrow y \in B_{sm}(x,r) \subseteq V \Rightarrow \underline{B}_{sm}(x,r/2) \subseteq V$

$\exists r>0 : \underline{B}_{sm}(x,r) \subseteq V$; $B_{sm}(x,r) \subseteq \underline{B}_{sm}(x,r)$; $B_{sm}(x,r) \in \tau_{sm}$

$\Rightarrow \exists M:=B_{sm}(x,r) \in \tau_{sm} : x \in B_{sm}(x,r) \subseteq V \Rightarrow V \in \mathcal{V}(x)$

C.3.4 s.f.v. de bile B,B̲

(X,sm) sp. sm.; $x \in X$;

$\{B(x,r), r>0\}$, $\{\underline{B}(x,r), r>0\}$ s.f.v. x

Def. Limită în sp. t.

(X,τ) sp. t.; $x \in X$, $(x_i)_{i\geq 1} \in (X)$;

$(x_i)_{i\geq 1} \to x \Leftrightarrow \forall V \in V(x) \; \exists i_0 : \forall i \geq i_0 \Rightarrow x_i \in V$

C.3.5 Limită în sp. sm.

(X,sm) sp. sm.; u.a.s.e.:

(1) $(x_i) \to x$;

(2) $\forall \varepsilon>0 \; \exists i_0 \geq 1 : \forall i > i_0 \; sm(x_i,x) < \varepsilon$;

(3) $(sm(x_i,x))_{i\geq 1} \to 0$;

C.3.6 Convergență şi limită în sp. sm.

(X,sm) sp. sm.; $(x_i)_{i\geq 1} \subseteq X$ convergent $\Leftrightarrow \exists x \in X : (x_i)_{i\geq 1} \to x$

$(x_i)_{i\geq 1} \subseteq X$, $\exists! x \in X : (x_i)_{i\geq 1} \to x$ *notăm* $x = \lim x_i$

(X,m) sp. m., $(x_i)_{i\geq 1} \subseteq X$, $x \in X$, $(x_i)_{i\geq 1} \to x \Rightarrow x = \lim x_i$

Aderență în sm

(X,sm) sp. sm.; $M \subseteq X$, $x \in X$;

Def. $x \in clM \Leftrightarrow \forall V \in \mathcal{V}(x) : V \cap M \neq \emptyset$;

P.3.7 Aderența unei mulțimi în sp. sm.

$x \in clM$ (aderența lui M) $\Leftrightarrow \exists (x_i) \subseteq M : (x_i) \to x$;

Cauchy

(X,sm) sp. sm.; $(x_i)_{i\geq 1} \subseteq X$ Cauchy (fundamental) $\Leftrightarrow$

$\forall \varepsilon>0, \exists i_0 \geq 1 : \forall n,m \geq i_0, sm(x_m,x_n) < \varepsilon$

P.3.8 Proprietățile şirurilor Cauchy în sp. sm.

(X,sm) sp. sm.; $(x_i)_{i\geq 1}\subseteq X$ u.a.s.e.:

(1) $(x_i)_{i\geq 1}$ Cauchy

(2) $\forall\varepsilon>0\ \exists i_0\geq 1 : \forall n\geq i_0, \forall k\geq 0 : sm(x_{n+k},x_n)<\varepsilon$

(3) $\forall\varepsilon>0\ \exists i\geq 1 : \forall n\geq i : sm(x_n,x_i)<\varepsilon$

P. 3.9. Convergență *vs.* Cauchy în sp. sm.

(X,sm) sp. sm.;$(x_i)_{i\geq 1}\subseteq X$; u.a.s.a.: (1) $(x_i)_{i\geq 1}$ convergent$\Rightarrow(x_i)_{i\geq 1}$ Cauchy;

(2) $(x_i)_{i\geq 1}$ Cauchy; $(x_i)_{i\geq 1}$ are subşir conv.$\Rightarrow(x_i)_{i\geq 1}$ conv.;

Mulțimi complete

(X,sm) sp. sm.; $M\subseteq X$ completă $\Leftrightarrow$

$\forall\ (x_i)_{i\geq 1}\subseteq M$ Cauchy $\Rightarrow \exists\ x\in M$: $x_i\rightarrow x$

Spațiu semimetric complet

(X,sm) sp. sm.; X completă $\Leftrightarrow$ (X,sm) sp. sm. complet;

P.3.10 Complet *vs.* închis

$(X,\Re,+,\cdot)$ sp. m.; $M\subseteq X$; u.a.s.a.: (1) M completă $\Leftrightarrow$ M închisă;

(2) M închisă; X completă $\Rightarrow$ M completă;

Operatori liniari

$(X,K,+,\cdot)$ s.l.; $(Y,K,+,\cdot)$ s.l.; $A:X\rightarrow Y$ operator;

(Aditiv) $\forall x_1,x_2\in X \Rightarrow A(x_1+x_2)=A(x_1)+A(x_2)$;

(Omogen) $\forall\alpha\in K, \forall x\in X \Rightarrow A(\alpha\cdot x)=\alpha\cdot A(x)$;

(Lin) $\forall\alpha_1,\alpha_2\in K,\forall x_1,x_2\in X \Rightarrow A(\Sigma\alpha_i\cdot x_i)=\Sigma\alpha_i\cdot A(x_i)$;

(Lin) $\Leftrightarrow$ (Aditiv)+(Omogen);

$L(X,Y)=\{ A:X\rightarrow Y$, A (Liniar)$\}$; $F(X,Y)=\{A:X\rightarrow Y\}$;

$L(X,Y)\subseteq F(X,Y)(\leq)$;

Izomorfisme algebrice

$(X,K,+,\cdot)$ s.l.; $(Y,K,+,\cdot)$ s.l.; $A:X\rightarrow Y$ operator;

A Izomorfism Algebric (IA) $\Leftrightarrow$ A (Lin) şi (Bijectiv);

$A:X\rightarrow Y$ (IA) $\Leftrightarrow$ X şi Y algebric izomorfe;

P.4.1 $(X,K,+,\cdot)$ s.l.; $(Y,K,+,\cdot)$ s.l.; $(Z,K,+,\cdot)$ s.l.; u.a.s.a.:

(1) $A:X\rightarrow Y$, $B:Y\rightarrow Z$ (Lin) $\Rightarrow B\circ A$ (Lin);

(2) $A:X\rightarrow Y$, $B:Y\rightarrow Z$ (IA) $\Rightarrow B\circ A$ (IA);

P.4.2 $(X,K,+,\cdot)$ s.l.; $(Y,K,+,\cdot)$ s.l.; $A:X\rightarrow Y$ (IA) $\Rightarrow A^{-1}:Y\rightarrow X$ (IA);

P.4.3 $(X,K,+,\cdot)$s.l.;$(Y,K,+,\cdot)$s.l.; X(I.A.)Y$\Leftrightarrow$dimX=dimY;

Funcționale liniare

$(X,K,+,\cdot)$s.l.;$(K,K,+,\cdot)$s.l.; $L(X,K)=\{A:X\rightarrow K$ (Lin)$\}$;

L(X,K) se numeşte *dualul algebric* al lui X;

f∈L(X,K) ⇒ f:X→K (Lin) – *funtională liniară*;

Ex.Hadamard

a,b∈ℜ; a<b; X:={x∈F([a,b],ℜ) | x integr. Riemann}

(X,ℜ,+,·)≤(F([a,b],ℜ),ℜ,+,·)

f:X→R; ∀x∈X f(x):=$\int_{[a,b]}x(t)dt$; f funcțională liniară;

f(x) funcție de funcții = funcțională (1903,Hadamard)

Funcție subliniară

(X,ℜ,+,·) s.l.; (ℜ,ℜ,+,·) s.l.; f:X→R subliniară ⇔

$\forall\alpha_1,\alpha_2\geq 0, \forall x_1,x_2\in\Re : f(\alpha_1x_1+\alpha_2x_2)\leq\alpha_1f(x_1)+\alpha_2f(x_2)$

Obs. $f(\alpha_1x_1+\alpha_2x_2)\leq\alpha_1f(x_1)+\alpha_2f(x_2)$ ⇔

(FSL1) $f(x_1+x_2)\leq f(x_1)+f(x_2)$ şi (FSL2) $f(\alpha x)=\alpha f(x)$;

(FSL1) ⇔ f SubAditivă; (FSL2) ⇔ f Pozitiv Omogenă;

(FSL2)⇒$f(O_X)=0$ ⇒ ∀f subliniară se anulează în O_X;

Ex.Max

$(\Re^m,\Re,+,\cdot)$s.l.;(ℜ,ℜ,+,·)s.l.;$f((x_1,\ldots,x_m))=\max\{x_1,\ldots,x_m\}$

f=max:$\Re^m\to\Re$ subliniară; f nu e funcțională liniară

Seminormă sn.

(X,ℜ,+,·) s.l.; f:X→ℜ sn. ⇔ $f(\alpha_1x_1+\alpha_2x_2)\leq|\alpha_1f(x_1)|+|\alpha_2f(x_2)|$

P.4.4 (X,ℜ,+,·) s.l.; f:X→ℜ sn. ⇔

(SN1) $f(x_1+x_2)\leq f(x_1)+f(x_2)$ şi (SN2) $f(\alpha x)=|\alpha f(x)|$;

(SN1) ⇔ f SubAditivă; (SN2) ⇔ f Absolut Omogenă;

P.4.5 Proprietățile sn. (X,ℜ,+,·) s.l.; f:X→ℜ sn.; u.a.s.a.:

(1) $\forall x_1,x_2\in X : |f(x_1)-f(x_2)|\leq f(x_1-x_2)$; (2) ∀x∈X ⇒ f(x)≥0

Normă

(X,ℜ,+,·) s.l.; f:X→ℜ normă ⇔ f:X→ℜ sn. şi (f(x)=0 ⇔ x=O_X)

L.5.1 Prelungirea funcționalelor liniare (Kelly Eduard)

(X,ℜ,+,·)s.l.; p:X→ℜ subliniară; (U,ℜ,+,·)≤(X,ℜ,+,·); f:U→ℜ (Lin); şi (1) ∀x∈U: f(x)≤p(x); Fie $x_0\in C_XU$; V:=U+lin{x_0} ⇒

⇒ ∃ g:V→U (Lin): $g|_U$=f şi (2) ∀x∈V g(x)≤p(x)

Obs. U⊆V; (V,ℜ,+,·) s.l.;

Relații binare

Fie X m.; R⊆X×X; x_1Rx_2 ⇔ $(x_1,x_2)\subseteq R$; R relație binară (re) R $\Leftrightarrow \forall x\in X \Rightarrow xRx$;

(tz) R $\Leftrightarrow \forall x,y,z\in X$, xRy; yRz $\Rightarrow$ xRz;

(si) R $\Leftrightarrow \forall x,y\in X$, xRy $\Rightarrow$ yRx;

(as) R $\Leftrightarrow \forall x,y\in X$, xRy, yRx $\Rightarrow$ x=y;

(ord) "$\leq$" $\Leftrightarrow$ (re) "$\leq$", (tz) "$\leq$", (as) "$\leq$";

(ech) "$\equiv$" $\Leftrightarrow$ (re) "$\equiv$", (tz) "$\equiv$", (si) "$\equiv$";

Mulțimi ordonate

Fie X m.; "$\leq$" (ord) pe X $\Leftrightarrow$ $(X,\leq)$ m.o.;

Fie $(X,\leq)$ m.o.; Fie $x_0\in X$; $M\subseteq X$;

(maj) $x_0\in X$ majorant M $\Leftrightarrow x_0\in maj(M) \Leftrightarrow \forall x\in M \Rightarrow x\leq x_0$

(max) $x_0\in max(M) \Leftrightarrow \urcorner\exists x\in M\setminus\{x_0\} : x_0\leq x,\ x_0\in M$;

(tord) M total ordonată $\Leftrightarrow \forall x_1,x_2\in M \Rightarrow x_1\leq x_2$ sau $x_2\leq x_1$;

(Maj) M Majorată $\Leftrightarrow maj(M)\neq\emptyset$;

(IndOrd) M Inductiv Ordonată $\Leftrightarrow \forall M\subseteq X$ (tord)$\Rightarrow$(Maj);

L.5.2 Principiul elementelor maximale

(?)

T.5.3 Prelungirea funcționalelor lin. reale - Hann H. și Banach S.

$(X,\Re,+,\cdot)$s.l.; $p:X\rightarrow\Re$ subliniară; $(W,\Re,+,\cdot)\leq(X,\Re,+,\cdot)$;

Fie $f_0:X_0\rightarrow\Re$ (Lin) cu (7) $\forall x\in W : f_0(x)\leq p(x) \Rightarrow$

$\exists\ f:X\rightarrow\Re : f|_W=f_0$ și (8) $\forall x\in X : f(x)\leq p(x)$

Dem. (Hann H. și Banach S.)

$P:=\{(U,f) \mid (U,\Re,+,\cdot)\leq(X,\Re,+,\cdot)$ și $f:U\rightarrow\Re$ (Liniară) cu (7)$\}$;

$(W,f_0)\in P \Rightarrow P\neq\emptyset$;

fie "$\leq$" : $(U,f),(V,g)\in P$; $(U,f)\leq(V,g) \Leftrightarrow U\subseteq V$ și $g|_U=f$; "$\leq$"(ord):

$(U,f)\leq(U,f)$; $(U,f)\leq(V,g)$; $(V,g)\leq(W,h) \Rightarrow (U,f)\leq(W,h)$;

$(U,f)\leq(V,g)$; $(V,g)\leq(U,f) \Rightarrow U=V$; $g=f \Rightarrow (U,f)=(V,g)$;

(?) $(P,\leq)$ (IndOrd) $\Leftrightarrow$ T(tord)$\subseteq$P $\exists x_0\in P : \forall x\in T \Rightarrow x\leq x_0$;

Dem.$(P,\leq)$ IndOrd.

Fie $T\subseteq P$ (tord); $T=\emptyset$ are majorant; $T\neq\emptyset$: fie Y :

$Y:=\{x\in X \mid \exists(U,f)\in T : x\in U\}$; $(U,f)\in T\neq\emptyset$; $U\neq\emptyset \Rightarrow Y\neq\emptyset$

Fie $h:Y\rightarrow\Re : h(y):=f(y)$; "h" e corect def. dacă ne duce pt. elemente y provenite de la diferiți U la aceeasi valoare f(y) în codomeniul $\Re$;

fie $(U_1,f_1),(U_2,f_2)\in T \Rightarrow (U_1,f_1)\leq(U_2,f_2)$ sau $(U_2,f_2)\leq(U_1,f_1)$

$\Rightarrow f_2|_{U1}=f_1$ sau $f_1|_{U2}=f_2 \Rightarrow (y\in U_1\cap U_2 \Rightarrow f_1(y)=f_2(y))$ c.definită;

(?) $(Y,h)\in P \Leftrightarrow (Y,\Re,+,\cdot)\leq(X,\Re,+,\cdot)$ şi $h:Y\to\Re$ (Liniară) cu (7)

$\forall y\in Y$, $f(y)=h(y)\leq p(y) \Rightarrow$ (7); (?) $Y:=\{x\in X \mid \exists(U,f)\in T : x\in U\}\leq X$;

Fie $y_1,y_2\in Y \Rightarrow \exists(U_1,f_1),(U_2,f_2)\in T$, $y_1\in U_1$, $y_2\in U_2$,

$(U_1,f_1)\leq(U_2,f_2)$ sau $(U_2,f_2)\leq(U_1,f_1)$; Fie $U_1\subseteq U_2 \Rightarrow y_1\in U_2$

Fie $\alpha_1,\alpha_2\in\Re$; $(U_2,\Re,+,\cdot)\leq(X,\Re,+,\cdot) \Rightarrow \alpha_1y_1+\alpha_2y_2\in U_2 \Rightarrow$

$\exists(U,f)=(U_2,f_2)\in T : \alpha_1y_1+\alpha_2y_2\in U \Rightarrow (Y,\Re,+,\cdot)\leq(X,\Re,+,\cdot)$;

(?) $h:Y\to\Re$ (Liniară) $h(y):=f_2(y)$ (Liniară) $\Rightarrow (Y,h)\in P$;

Fie $(U,f)\in T$, $h|_U=f \Rightarrow (U,f)\leq(Y,h)$ în $P \Rightarrow (Y,h)\in maj(T) \Rightarrow$

$(P,\leq)$ (IndOrd); $(X_0,f_0)\in P \Rightarrow \exists(Z,i)\in P : (X_0,f_0)\leq(Z,i)$;

$Z\neq X$, $Z\subseteq X \Rightarrow \exists x_0\in C_XZ$; $(Z,i)\in P$; $V:=Z+lin\{x_0\}$ (K.E.) $\Rightarrow$

$\exists g:V\to\Re$ (Liniară) cu (7) $\Rightarrow (V,g)\in P$; $Z\subseteq V$; $V\leq Z \Rightarrow V=Z$ (F)

$\Rightarrow Z=X \Rightarrow (X,f)\in P \Rightarrow \forall(X_0,f_0)\in P : (X_0,f_0)\leq(X,f) \Rightarrow f|_{X0}=f$;

Ex. Metrică în F(T,K)

$T\neq\emptyset$; $F(T,K)$ sp.l.; $p:F(T,K)\to\Re\cup\{\infty\}$;

$p(x):=\sup\{|x(t)|,t\in T\}$

(1) $d:F(T,K)\times F(T,K)\to\Re$, $d(x,y):=\min\{1,p(x-y)\}$ metrică

(2) $(F(T,K),d)$ sp.m. complet; (3) $\tau_d=\tau_u$

Rez. (1) $\forall x\in F(T,K)$ $d(x,x)=\min\{1,p(x-x)\}=\min\{1,0\}=0$;

$d(x,y)=\min\{1,p(x-y)\}= \min\{1,p(x-y)\}=d(y,x)$

Fie $x,y,z\in F(T,K)$ (?) $d(x,y)\leq d(x,z)+d(z,y)$

(a) $p(x-z)\geq1$ sau $p(z-y)\geq1 \Rightarrow d(x,z)=1$ sau $d(z,y)=1 \Rightarrow$

$d(x,z)+d(z,y)\geq1\geq d(x,y)$

(b) $p(x-z)<1$ şi $p(z-y)<1 \Rightarrow d(x,z)=p(x-z)$ şi $d(z,y)=p(z-y)$

$\forall t\in T$ $|x(t)-y(t)|\leq|x(t)-z(t)|+|z(t)-y(t)|\leq p(x-z)+p(z-y) \Rightarrow$

$d(x,y)\leq p(x-y)\leq p(x-z)+p(z-y)=d(x,z)+d(z,y)$

Fie $x,y\in F(T,K)$ cu $x\neq y \Rightarrow \exists t_0\in T : x(t_0)\neq y(t_0) \Rightarrow$

$p(x-y)=\sup\{|x(t)-y(t)|,t\in T\}\geq|x(t_0)-y(t_0)|>0 \Rightarrow$

$d(x,y)=\min\{1,p(x,y)\}\geq\min\{1,|x(t_0)-y(t_0)|\}>0 \Rightarrow d(x,y)>0$

(2) Fie $(x_i)_{i\geq1}\subseteq F(T,K)$ Cauchy, $t\in T$, $\varepsilon>0$; $\varepsilon':=\min\{\varepsilon,1\}$;

$(x_i)_{i\geq1}$ Cauchy $\Rightarrow \exists i_0\geq1 : \forall m,n\geq i_0$ $d(x_m,x_n)<\varepsilon'\leq1 \Rightarrow$

$\forall m,n\geq i_0$ $d(x_m,x_n)=p(x_m,x_n)<\varepsilon'\leq\varepsilon \Rightarrow |x_m(t)-x_n(t)|<\varepsilon \Rightarrow$

$(x_i(t))_{i\geq1}$ Cauchy în K; (K,d) complet $\Rightarrow \exists x\in K : x_i(t)\to x$

(3) $x_0 \in F(T,K)$; $B_T(f_0,r_0)=\{f\in F(T,K) : \forall t\in T\ |f(t)-f_0(t)|<r_0\}$

$U:F(T,K)\to \wp(\wp(F(T,K)))$; $U(f)=\{B_T(f,r), 0<r<\infty\}$

$(F(T,K),\tau_U)$ sp.top.; $\forall x\in F(T,K)$, $U(x)$ s.f.v. a lui x, $U(x)$ unic

$\tau_d=\tau_U \Leftrightarrow U(x)$ s.f.v. a lui x în $(F(T,K),\tau_d)$

Ex. Metrica unei serii

(1) $(x_i)_{i\geq 1},(y_i)_{i\geq 1}\in(K)$; $s_i=\Sigma_{1\leq j\leq i}(1/2^j)\cdot|x_j-y_j|/(1+|x_j-y_j|)$ conv.;

(2) $d:(K)\times(K)\to\Re$, $d(x,y)= \Sigma_{i\geq 1}(1/2^i)\cdot|x_i-y_i|/(1+|x_i-y_i|)$ m.;

(3) $((K),d)$ sp. m. complet;

<u>dem.</u>(1) $\Sigma(1/2^j)$ conv.; $0\leq(1/2^j)\cdot|x_j-y_j|/(1+|x_j-y_j|)\leq(1/2^j) \Rightarrow$

$\Sigma(1/2^j)\cdot|x_j-y_j|/(1+|x_j-y_j|)$ conv.

(2) $d(x,y) = 0 \Leftrightarrow (1/2^i)\cdot|x_i-y_i|/(1+|x_i-y_i|)=0 \Leftrightarrow x_i=y_i \Leftrightarrow x=y$

$d(x,y)=\Sigma(1/2^i)\cdot|x_i-y_i|/(1+|x_i-y_i|)=\Sigma(1/2^i)\cdot|y_i-x_i|/(1+|y_i-x_i|)=d(y,x)$

$d(x,y)\leq d(x,z)+d(z,y) \Leftrightarrow \Sigma(1/2^i)\cdot|x_i-y_i|/(1+|x_i-y_i|) \leq$

$\Sigma(1/2^i)\cdot|x_i-z_i|/(1+|x_i-z_i|)+\Sigma(1/2^i)\cdot|z_i-y_i|/(1+|z_i-y_i|) \Leftrightarrow$

$\Sigma(1/2^i)\cdot(|x_i-z_i|/(1+|x_i-z_i|)+|z_i-y_i|/(1+|z_i-y_i|)-|x_i-y_i|/(1+|x_i-y_i|))\geq 0$

$\Leftarrow |x_i-z_i|/(1+|x_i-z_i|)+|z_i-y_i|/(1+|z_i-y_i|) \geq |x_i-y_i|/(1+|x_i-y_i|)$ (a)

$u:=x_i-z_i$; $v:=z_i-y_i$; a $\Leftrightarrow |u+v|/(1+|u+v|)\leq|u|/(1+|u|)+|v|/(1+|v|)$

$\Leftrightarrow |u+v|(1+|u|)(1+|v|)\leq|u|(1+|v|)(1+|u+v|)+|v|(1+|u|)(1+|u|+|v|)$

$|u+v|+(|u|+|v|)|u+v|+|u+v||u||v|\leq|u|+|u||v|+|u||u+v|+|u+v||u||v|+$

$|v|+|u||v|+|v||u+v|+|u+v||u||v| \Leftrightarrow |u+v|\leq|u|+|v|+2|u||v|+|u+v||u||v|$

dar $|u+v|\leq|u|+|v|\leq|u+v|\leq|u|+|v|2|u||v|+|u+v||u||v|$;

(?) $((K),d)$ complet; $x_n:=(x_{n,1},\ldots)$; x_n fundamental în (K)

$\Rightarrow\forall\varepsilon>0\ \exists n_0 : \forall n,m\geq n_0 : d(x_n,x_m)<\varepsilon \Rightarrow$

$\forall\varepsilon>0\ \exists n_0 : \forall n,m\geq n_0 : \Sigma_{i\geq 1}(1/2^i)\cdot|x_{n,i}-x_{m,i}|/(1+|x_{n,i}-x_{m,i}|)<\varepsilon$;

$\Rightarrow \Sigma_{1\leq i\leq k}<\varepsilon$; $2^k\varepsilon>2^k\Sigma_{1\leq i\leq k}(1/2^i)\cdot|x_{n,i}-x_{m,i}|/(1+|x_{n,i}-x_{m,i}|)=$

$\Sigma_{1\leq i\leq k}2^{k-i}|x_{n,i}-x_{m,i}|/(1+|x_{n,i}-x_{m,i}|)\geq\Sigma_{1\leq i\leq k}|x_{n,i}-x_{m,i}|/(1+|x_{n,i}-x_{m,i}|)\geq$

$\Sigma_{1\leq i\leq k}|x_{n,i}-x_{m,i}|\geq|x_{n,k}-x_{m,k}|$; $\forall k\geq 0$ avem că $\forall\varepsilon'>0$, $\exists m_0\geq 0$ a.î.

$\forall n,m\geq m_0 : |x_{n,k}-x_{m,k}|<\varepsilon'$ unde $m_0=m_0(\varepsilon'):=n_0(2^k\varepsilon) \Rightarrow$

$\forall k$, $(x_{i,k})_{i\geq 0}$ fundamental în K $\Rightarrow \forall k$, $(x_{i,k})_{i\geq 0}$ convergent în K $\Rightarrow$

$x_i=(x_{i,1},\ldots x_{i,k},\ldots)$; x_i convergent în (K);

Topologie compatibilă cu structura algebrică a unui s.l.

$(X,K,+,\cdot)$ s.l.; $\tau\subseteq\wp(X)$ a.î. (X,τ) sp.t.;

(TL1) $\forall x,y\in X \Rightarrow (x+y)\in X$; (TL2) $\forall x\in X, \forall\alpha\in K \Rightarrow \alpha x\in X$

τ compatibilă cu $(+,\cdot) \Leftrightarrow$ (TL1$\Leftrightarrow$) şi (TL2$\Leftrightarrow$);

(TL1$\Leftrightarrow$) $\forall x_0,y_0 \in X$, $\forall W \in V(x_0+y_0)$, $\exists U \in V(x_0)$, $\exists V \in V(y_0)$:

$\forall x \in U$, $\forall y \in V \Rightarrow x+y \in W$;

(TL2$\Leftrightarrow$) $\forall \alpha_0 \in K$, $\forall x_0 \in X$, $\forall W \in V(\alpha_0 x_0)$ $\exists U \in V(\alpha_0)$, $\exists V \in V(x_0)$:

$\forall \alpha \in U$, $\forall x \in V \Rightarrow \alpha x \in W$;

Def. $(X,K,+,\cdot)$ s.l.; (X,τ) sp.t.; (TL1); (TL2) $\Rightarrow$ $(X,K,+,\cdot,\tau)$ s.l.t.

Def. Continuitatea operatorilor "+" şi "·" în s.l.t.

$(X,K,+,\cdot,\tau)$ s.l.t.;

(TL1$\Leftrightarrow$) : (cont. "+" în $X\times X$); (TL2$\Leftrightarrow$) : (cont. "·" în $K\times X$);

Ex. F(T,K) s.l.t.

$(F(T,K),K,+,\cdot)$ s.l.; $p:F(T,K)\to\Re\cup\{\infty\}$; $p(x):=\sup\{|x(t)|,t\in T\}$;

$d:F(T,K)\times F(T,K)\to\Re$; $d_p(x,y)=\min\{1,p(x-y)\}$;

$B_T(x_0,r_0)=\{f\in F(T,K), \forall t\in T \Rightarrow |x(t)-x_0(t)|<r_0\}$;

$U:F(T,K)\to \wp(\wp(F(T,K)))$; $U(x):=\{B_T(x,r), r>0\}$;

$\tau_U:=\{M\subseteq F(T,K), \forall x\in F(T,K) \Rightarrow M\in U(x)\}$;

d metrică pe F(T,K); τ_U topologie pe F(T,K); τ_U topologia uniformă;

$U(x)$ s.f.v. x $\Rightarrow \tau_U$ unică a.î. $\forall x\in F(T,K)$, $U(x)=V(x)$;

$\Rightarrow (F(T,K),K,+,\cdot,\tau_U)$ s.l.t.

Ex. Topologii vs. structuri algebrice

$(X,K,+,\cdot)$ s.l.; $X\neq\{O_X\}$; $\tau_{indis}=\{\emptyset,X\}$; $\tau_{dis}=\wp(X)$;

(τ_{indis}) $\forall x\in X$, $V(x)=\{X\}$;

(TL1$\Leftrightarrow$) : $\forall x_0,y_0\in X$, $W=X$, $\exists U=V=X$, $\forall x,y\in X$ $x+y\in X$ cf. (TL1)

(TL2$\Leftrightarrow$) : $\forall \alpha_0\in K$, $\forall x_0\in X$, $W=X$, $\exists U=K$, $\exists V=X$: $\forall\alpha\in K$, $\forall x\in X$

$\Rightarrow \alpha x\in X$ cf. (TL2)

$(\tau_{dis}=\wp(X))$; $\forall x\in X$, $V(x)=\{V\subseteq X, x\in V\}$;

(TL1$\Leftrightarrow$) : Fie $x_0,y_0\in X$; $W\in V(x_0+y_0)$, $\exists U:=\{x_0\}\in V(x_0)$,

$\exists V:=\{y_0\}\in V(y_0)$ $\forall x=x_0\in U$ $\forall y=y_0\in V \Rightarrow x+y=x_0+y_0\in W$ evident;

$(0,O_X)\in K\times X$; $(X,K,+,\cdot)$ s.l. $\Rightarrow \forall x\in X : 0\cdot x=O_X$;

(?⌉TL2$\Leftrightarrow$) $\forall \alpha_0 \in K$, $\forall x_0 \in X$, $\forall W \in V(\alpha_0 x_0)$ $\exists U \in V(\alpha_0)$, $\exists V \in V(x_0)$:

$\forall \alpha \in U$, $\forall x \in V \Rightarrow \alpha x \in W$;

Fie $x_0\in C_X\{O_X\}$; $\alpha_0:=0$; $W:=\{O_X\}$; Fie $U\in V(0)$; Fie $\alpha:=1\in U\setminus\{0\}$;

Fie $x:=x_0\in V\in V(x_0) \Rightarrow \alpha x=1\cdot x_0=x_0\in\{O_X\} \Rightarrow x_0=O_X$ contrad. $\Rightarrow$

$\Rightarrow$ (TL2$\Leftrightarrow$) nu are loc $\Rightarrow \wp(X)$ nu e compatibilă cu str. alg. X.

T.7.1 Topologia unei sm. pe un s.l.

$(X,K,+,\cdot)$ s.l.; $\|\cdot\|:X\to\Re$ (SubAditivă);

$\exists 0<k\leq 1 : \forall\alpha\in K, \forall x\in X\ \|\alpha\cdot x\|=|\alpha|^k\cdot\|x\|$; $d:X\times X\to\Re$, $d(x,y)=\|x-y\|$

$\Rightarrow (X,d)$ sp.sm.; τ_d comp. cu str. alg. X;

Dem.

(SM1) $d(x,x)=\|x-x\|=\|O_X\|=\|0\cdot O_X\|=|0|^k\cdot\|O_X\|=0$

(SM2) $d(x,y)=\|x-y\|=\|(-1)\cdot(y-x)\|=|-1|^k\cdot\|y-x\|=\|y-x\|=d(y,x)$

(SM3) $d(x,z)+d(z,y)=\|x-z\|+\|z-y\|\geq$(S.A.)$\geq\|x-y\|=d(x,y)$

$\tau_d:=\{M\subseteq X \mid \forall x\in M\ \exists r>0 : B_d(x,r)\subseteq M\}$;

$B_d(x_0,r_0)=\{x\in X : d(x,x_0)<r_0\}$; $\{B_d(x,r), r>0\}$ s.f.v. x;

(?TL1$\Leftrightarrow$) $\forall x_0,y_0\in X, \forall W\in V(x_0+y_0), \exists U\in V(x_0), \exists V\in V(y_0)$:

$\forall x\in U, \forall y\in V \Rightarrow x+y\in W$;

(?TL2$\Leftrightarrow$) $\forall\alpha_0\in K, \forall x_0\in X, \forall W\in V(\alpha_0 x_0)\ \exists U\in V(\alpha_0), \exists V\in V(x_0)$:

$\forall\alpha\in U, \forall x\in V \Rightarrow \alpha x\in W$;

$x_0,y_0\in X$; $W\in V(x_0+y_0) \Rightarrow \exists r_0\in\Re : B_d(x_0+y_0,r_0)\subseteq W$;

$U:=B_d(x_0,r_0/2)\in V(x_0)$; $V:=B_d(y_0,r_0/2)\in V(y_0)$; fie $x\in U, y\in V \Rightarrow$

$d(x+y,x_0+y_0)=\|x+y-(x_0+y_0)\|=\|(x-x_0)+(y-y_0)\|\leq\|x-x_0\|+\|y-y_0\|=$

$=d(x,x_0)+d(y,y_0)<r_0/2+r_0/2=r_0 \Rightarrow (x,y)\in B_d(x_0+y_0,r_0)\subseteq W$ (TL1$\Leftrightarrow$)

$x_0\in X, \alpha_0\in K$; $W\in V(\alpha_0\cdot x_0) \Rightarrow \exists r_0\in\Re : B_d(\alpha_0\cdot x_0,r_0)\subseteq W$;

$2\cdot\|x_0\|=\|x_0\|+\|x_0\|\geq\|2x_0\|=2^k\|x_0\| \Rightarrow (2-2^k)\|x_0\|\geq 0$; $2-2^k\geq 0 \Rightarrow \|x_0\|\geq 0$

$r:=\min\{1,r_0/(1+|\alpha_0|^k+\|x_0\|)\}$; $U:=B(\alpha_0,r^{1/k})\in V(\alpha_0)$;

$V:=B_d(x_0,r)\in V(x_0)$; fie $\alpha\in U, x\in V \Rightarrow |\alpha-\alpha_0|<r$; $\|x-x_0\|<r$;

$d(\alpha x,\alpha_0 x_0)=\|\alpha x-\alpha_0 x_0\|=\|\alpha x-\alpha_0 x+\alpha_0 x-\alpha_0 x_0\|=\|(\alpha-\alpha_0)x+\alpha_0(x-x_0)\|=$

$=\|(\alpha-\alpha_0)x-(\alpha-\alpha_0)x_0+(\alpha-\alpha_0)x_0+\alpha_0(x-x_0)\|=$

$=\|(\alpha-\alpha_0)\cdot(x-x_0)+(\alpha-\alpha_0)x_0+\alpha_0(x-x_0)\|\leq\|(\alpha-\alpha_0)\cdot(x-x_0)\|+\|(\alpha-\alpha_0)x_0\|+$

$+\|\alpha_0(x-x_0)\|=|\alpha-\alpha_0|^k\cdot\|x-x_0\|+|\alpha-\alpha_0|^k\cdot\|x_0\|+|\alpha_0|^k\cdot\|x-x_0\|<$

$r^2+r\cdot\|x_0\|+r\cdot|\alpha_0|^k=r(r+\|x_0\|+|\alpha_0|^k)\leq r(1+|\alpha_0|^k+\|x_0\|)=(1+|\alpha_0|^k+\|x_0\|)r\leq$

$\leq(1+|\alpha_0|^k+\|x_0\|)r_0/(1+|\alpha_0|^k+\|x_0\|)=r_0 \Rightarrow \alpha x\in B_d(\alpha_0 x_0,r_0)\subseteq W$

Topologia unei sm. pe (K)

$((K),K,+,\cdot)$ s.l.; Fie $0<p<1$; $l_p=\{(\xi_i)_{i\geq 1}\subseteq(K) : \Sigma|\xi_i|^p<\infty\}$;

$(l_p,K,+,\cdot)\leq((K),K,+,\cdot) \Rightarrow (l_p,K,+,\cdot)$ s.l.; Fie $\|\cdot\|:l_p\to\Re$,

$\forall x=(x_k)_{k\geq 1}\in l_p$, $\|x\|:=\Sigma_{k\geq 1}|x_k|^p \Rightarrow \forall\alpha\in K, \forall x\in l_p$, $\|\alpha x\|=|\alpha|^p\|x\|$;

$\|x+y\|=\Sigma_{k\geq 1}|x_k+y_k|^p\leq\Sigma_{k\geq 1}|x_k|+\Sigma_{k\geq 1}|y_k|^p=\|x\|+\|y\| \Rightarrow \|\cdot\|$ (SubAd);

$\tau_{\|\cdot\|}=\{M\subseteq l_p \mid \forall x\in M\ \exists r>0 : B_{\|\cdot\|}(x,r)\subseteq M\}$;

$B_{\|\cdot\|}(x_0,r_0)=\{x\in l_p : \|x-x_0\|<r_0\} \Rightarrow (l_p,K,+,\cdot,\tau_{\|\cdot\|})$ s.l.t.;

$d:l_p\times l_p\to\Re$ $d(x,y):=\|x-y\|$; (?) d metrică: $d(x,y)=\|x-y\|=\Sigma_{k\geq1}|x_k-y_k|^p=0 \Leftrightarrow$

$\forall k\geq1\ |x_k-y_k|=0 \Leftrightarrow x=y$; $d(x,y)=\|x-y\|=\Sigma_{k\geq1}|x_k-y_k|^p=\Sigma_{k\geq1}|y_k-x_k|^p=d(y,x)$;

$d(x,z)+d(z,y)=\|x-z\|+\|z-y\|=\Sigma_{k\geq1}|x_k-z_k|^p+\Sigma_{k\geq1}|z_k-y_k|^p=$

$\Sigma_{k\geq1}(|x_k-z_k|^p+|z_k-y_k|^p)\geq\Sigma_{k\geq1}|x_k-y_k|^p=\|x-y\|=d(x-y) \Rightarrow$ d metrică

C.7.2 Convergență prin "+" și "·" în s.l.t.

$(X,K,+,\cdot,\tau)$ s.l.t.; Fie $(x_n)_{n\geq1},(y_n)_{n\geq1}\in(X)$, $x_n\to x_0$, $y_n\to y_0$; $x_0,y_0\in X$;

Fie $(\alpha_n)_{n\geq1}\in(K)$, $\alpha_n\to\alpha_0$, $\alpha_0\in K \Rightarrow (x_n+y_n)\to(x_0+y_0)$; $(\alpha_n x_n)\to(\alpha_0 x_0)$.

Dem.7.2 (?1) $\forall W\in V(x_0+y_0)$, $\exists k_0\geq1 : \forall i\geq k_0 \Rightarrow (x_i+y_i)\in W$

Fie $W\in V(x_0+y_0)$; $(X,K,+,\cdot,\tau)$ s.l.t. $\Rightarrow \exists U\in \mathcal{V}(x_0)$, $\exists V\in \mathcal{V}(y_0)$:

$\forall x\in U$, $\forall y\in V$, $(x+y)\in W$ (a);

$(x_i)_{i\geq1}\to x_0$, $U\in \mathcal{V}(x_0) \Rightarrow \exists i_0\geq1 : \forall i\geq i_0$, $x_i\in U$

$(y_i)_{i\geq1}\to y_0$, $V\in \mathcal{V}(y_0) \Rightarrow \exists j_0\geq1 : \forall i\geq j_0$, $y_i\in V$

$k_0:=\max\{i_0,j_0\} \Rightarrow \forall i\geq k_0$, $x_i\in U$, $y_i\in V$; (a) $\Rightarrow$

$\Rightarrow \forall i\geq k_0$, $(x_i+y_i)\in W\subseteq V(x_0+y_0) \Rightarrow (x_i+y_i)\to(x_0+y_0)$

(?2) $\forall W\in V(\alpha_0 x_0)$, $\exists k_0\geq1 : \forall i\geq k_0 \Rightarrow (\alpha_i x_i)\in W$

Fie $W\in V(\alpha_0 x_0)$; $(X,K,+,\cdot,\tau)$ s.l.t. $\Rightarrow \exists U\in \mathcal{V}(\alpha_0)$, $\exists V\in \mathcal{V}(x_0)$:

$\forall\alpha\in U$, $\forall x\in V$, $\alpha x\in W$ (b);

$(\alpha_i)_{i\geq1}\to\alpha_0$, $U\in \mathcal{V}(\alpha_0) \Rightarrow \exists i_0\geq1 : \forall i\geq i_0$, $\alpha_i\in U$

$(x_i)_{i\geq1}\to x_0$, $V\in \mathcal{V}(x_0) \Rightarrow \exists j_0\geq1 : \forall i\geq j_0$, $x_i\in V$

$k_0:=\max\{i_0,j_0\} \Rightarrow \forall i\geq k_0$, $\alpha_i\in U$, $x_i\in V$; (b) $\Rightarrow$

$\Rightarrow \forall i\geq k_0$, $(\alpha_i x_i)\in W\subseteq V(\alpha_0 x_0) \Rightarrow \alpha_i x_i\to\alpha_0 x_0$

P.7.1 Vecinătăți induse prin "+" și "·" în s.l.t.

$(X,+,\cdot,\tau)$ s.l.t.; $x_0\in X$; $V\in \mathcal{V}(x_0)$; u.a.s.a.:

(1) $\forall x\in X$, $x+V\in \mathcal{V}(x+x_0)$; (2) $\forall\alpha\in K$, $\alpha V\in \mathcal{V}(\alpha x_0)$;

Dem.7.1

$x_0=(x+x_0)+(-x)$; $(x+x_0),(-x)\in X \Rightarrow$

$\forall W\in \mathcal{V}(x_0)$, $\exists U\in \mathcal{V}(x+x_0)$, $\exists V\in \mathcal{V}(-x) : \forall u\in U$, $\forall v\in V$, $u+v\in W$

$v:=-x \Rightarrow \exists U\in \mathcal{V}(x+x_0) : \forall u\in U$, $u+(-x)\in W \Rightarrow$

$\forall W\in \mathcal{V}(x_0)$, $\exists U\in \mathcal{V}(x+x_0) : U-x\subseteq W \Rightarrow$

$\forall W\in \mathcal{V}(x_0)$, $\exists U\in \mathcal{V}(x+x_0) : U\subseteq x+W \Rightarrow (x+W)\in \mathcal{V}(x+x_0)$;

$W:=V \Rightarrow (x+V)\in \mathcal{V}(x+x_0)$

$\alpha\in K^* \Rightarrow x_0=(1/\alpha)\cdot\alpha x_0;\ (1/\alpha)\in K, (\alpha x_0)\in X \Rightarrow$

$\forall W\in \mathcal{V}(x_0), \exists U\in \mathcal{V}(\alpha x_0), \exists V\in \mathcal{V}(1/\alpha) : \forall u\in U, \forall v\in V, uv\in W$

$v:=1/\alpha \Rightarrow \exists U\in \mathcal{V}(\alpha x_0), \forall u\in U, (1/\alpha)u\in W \Rightarrow$

$\forall W\in \mathcal{V}(x_0), \exists U\in \mathcal{V}(\alpha x_0), (1/\alpha)U\subseteq W \Leftrightarrow$

$\forall W\in \mathcal{V}(x_0), \exists U\in \mathcal{V}(\alpha x_0), U\subseteq\alpha W \Rightarrow \forall W\in \mathcal{V}(x_0), \alpha W\in \mathcal{V}(\alpha x_0)$

$W:=V \Rightarrow \alpha V\in \mathcal{V}(\alpha x_0);$

$\alpha=0,\ 0V=\{x \mid x=0\cdot v=O_X, v\in V\}=\{O_X\}$ (?) $\{O_X\}\in \mathcal{V}(O_X)$

$\{O_X\}\in \mathcal{V}(O_X) \Leftrightarrow \{O_X\}\in\tau$ (?!)

C.7.2 S.f.v. indus prin "+" şi "·" în s.l.t.

$(X,K,+,\cdot,\tau)$ s.l.t.; $x_0\in X$; $\mathcal{U}(x_0)\subseteq \wp(X)$ s.f.v. x_0;

$\mathcal{U}(x+x_0):=\{x+U \mid U\in \mathcal{U}(x_0)\} \Rightarrow \mathcal{U}(x+x_0)$ s.f.v. $x+x_0$

Dem. $x\in X$; $U\in \mathcal{U}(x_0) \Rightarrow U\in \mathcal{V}(x_0)$; P.7.1 $\Rightarrow x+U\in \mathcal{V}(x+x_0) \Rightarrow$

$\mathcal{U}(x+x_0) \subseteq \mathcal{V}(x+x_0)$; Fie $V\in \mathcal{V}(x+x_0)$; P.7.1$\Rightarrow V-x \in \mathcal{V}(x_0) \Rightarrow \exists U\in \mathcal{U}(x_0) : U\subseteq V-x \Rightarrow$

$x+U\subseteq V\in \mathcal{V}(x+x_0) \Rightarrow \forall V\ \exists U : x+U\subseteq V;$

$U\in \mathcal{U}(x_0) \Rightarrow x+U\in \mathcal{U}(x+x_0) \Rightarrow \mathcal{U}(x+x_0)$ s.f.v. $x+x_0$

P.7.3 Int şi cl induse de "+" şi "·"

$(X,K,+,\cdot,\tau)$ s.l.t.; $M\subseteq X$; u.a.s.a.:

(1) $\forall x_0\in X,\ x_0+\text{int}M=\text{int}(x_0+M);\ x_0+\text{cl}M=\text{cl}(x_0+M)$

(2) $\forall\alpha\in K^*\ \text{int}(\alpha M)=\alpha\cdot\text{int}M;\ \text{cl}(\alpha M)=\alpha\cdot\text{cl}M;$

Dem. $(x\in\text{Int}M \Leftrightarrow M\in \mathcal{V}(x))$; $(x\in\text{cl}M \Leftrightarrow \forall V\in \mathcal{V}(x) : V\cap M\neq\varnothing)$;

$(x_0+A=\{x\in X \mid \exists a\in A : x=x_0+a\})$;

D.1: fie $x\in x_0+\text{int}M \Rightarrow \exists a\in\text{int}M$: $x=x_0+a$; $M\in \mathcal{V}(a)$; P.7.1 $\Rightarrow x_0+M\in \mathcal{V}(x_0+a) \Rightarrow x=x_0+a\in\text{Int}(x_0+M)$

$\Rightarrow x_0+\text{int}M\subseteq \text{Int}(x_0+M);$

fie $x\in\text{int}(x_0+M) \Rightarrow x_0+M\in \mathcal{V}(x)$; P.7.1 $\Rightarrow M\in \mathcal{V}(x-x_0) \Rightarrow x-x_0\in\text{Int}M$

$\Rightarrow x\in x_0+\text{int}M \Rightarrow \text{Int}(x_0+M) \subseteq x_0+\text{int}M \Rightarrow \text{Int}(x_0+M) = x_0+\text{int}M;$

$x\in x_0+\text{cl}M \Rightarrow x-x_0\in\text{cl}M$; fie $V\in \mathcal{V}(x)$; P.7.1 $\Rightarrow V-x_0\in \mathcal{V}(x-x_0);$

$x-x_0\in\text{cl}M,\ V-x_0\in \mathcal{V}(x-x_0) \Rightarrow M\cap V-x_0\neq\varnothing \Rightarrow x_0+M\cap V\neq\varnothing \Rightarrow$

$x\in\text{cl}(x_0+M) \Rightarrow x_0+\text{cl}M\subseteq \text{cl}(x_0+M);$

$x\in\text{cl}(x_0+M)$; fie $V\in \mathcal{V}(x-x_0)$; P.7.1 $\Rightarrow x_0+V\in \mathcal{V}(x)$; $M+x_0\cap V+x_0\neq\varnothing$

$\Rightarrow V\cap M\neq\varnothing \Rightarrow x-x_0\in\text{cl}M \Rightarrow x\in x_0+\text{cl}M \Rightarrow \text{cl}(x_0+M) \subseteq x_0+\text{cl}M;$

D.2: fie $x\in\text{int}(\alpha M) \Rightarrow \alpha M\in \mathcal{V}(x)$; P.7.3.1 $\Rightarrow M\in V(\alpha^{-1}x) \Rightarrow$

$\alpha^{-1}x\in\text{Int}M \Rightarrow x\in\alpha\text{Int}M \Rightarrow \text{int}(\alpha M)\subseteq\alpha\text{Int}M;$

fie $x\in\alpha\text{Int}M \Rightarrow \alpha^{-1}x\in\text{Int}M \Rightarrow M\in V(\alpha^{-1}x)$; P.7.3.1 $\Rightarrow \alpha M\in V(x)$

$\Rightarrow x\in int(\alpha M) \Rightarrow \alpha IntM\subseteq int(\alpha M) \Rightarrow \alpha IntM = int(\alpha M)$;

P.7.4 Acoperire şi închidere induse de "+" şi "·" în s.l.t.

$(X,K,+,\cdot,\tau)$ s.l.t.; $M\subseteq X$; $x_0\in X$, $\alpha\in K$; u.a.s.a.:

(1) $cl(x_0+M)=x_0+cl(M)$; $cl(\alpha M)=\alpha\cdot cl(M)$;

(2) M închisă $\Rightarrow x_0+M$, αM închise;

Categorii topologice

(T,τ) sp.t.; $M\subseteq \wp(T)$;

Def1. M nicăieri densă $\Leftrightarrow Int(cl(M))=\emptyset$;

Def2. M de categ. I $\Leftrightarrow \exists(M_n)_{n\geq1}\subseteq \wp(T)$, M_n nicăieri dense : $M=\cup M_n$

Def3. M de categ. II $\Leftrightarrow$ M nu este de categ. I;

Def4. (T,τ) sp.t. Baire $\Leftrightarrow \forall M\in\tau\setminus\emptyset \Rightarrow$ M este de categ. II;

T.7.5 Baire *vs.* categ. II

$(X,+,\cdot,\tau)$ s.l.t.; $(X,+,\cdot,\tau)$ Baire $\Leftrightarrow$ X de categ. II;

Dem.7.5 "$\Rightarrow$" evidentă din definiție pt. M:=X; "$\Leftarrow$":

Fie $M\in\tau\setminus\emptyset$; pp. că M de categ. I $\Rightarrow$

$\Rightarrow \exists(M_n)_{n\geq1}\subseteq \wp(X)$, M_n nicăieri dense : $M=\cup_{n\geq1}M_n$;

Fie $x_0\in M$; $m\geq1$; $N_m:=m(M-x_0)$;

$N_m=m(M-x_0)=m(\cup_{n\geq1}M_n-x_0)=m\cup_{n\geq1}(M_n-x_0)=\cup_{n\geq1}m(M_n-x_0)$;

$int(cl(m(M_n-x_0)))=m\cdot int(cl(M_n))+m\cdot x_0=m\cdot\emptyset+m\cdot x_0=\emptyset$

$\Rightarrow N_m$ de categ. I; (?) $X=\cup_{m\geq1}N_m$;

$x_0\in M\in\tau \Rightarrow M\in V(x_0)$; P.7.1 $(x=x_0) \Rightarrow M-x_0\in V(x_0-x_0)=V(O_X)$

Fie $x\in X$; $k^{-1}x\in X$; $k^{-1}x\to O_X \Rightarrow \exists k_0$: $\forall k\geq k_0$, $k^{-1}x\in M-x_0\in V(O_X) \Rightarrow$

$\exists k_0$: $\forall k\geq k_0$, $x\in k(M-x_0)\in V(O_X) \Rightarrow \exists k_0$: $\forall k\geq k_0$, $x\in N_k\subseteq\cup_{m\geq1}N_m \Rightarrow$

$X\subseteq\cup_{m\geq1}N_m$; $N_m\subseteq X \Rightarrow \cup_{m\geq1}N_m\subseteq X \Rightarrow X=\cup_{m\geq1}N_m \Rightarrow$ X de categ. I

(F) $\Rightarrow \forall M\in\tau\setminus\emptyset$, M este de categ. II;

Spații seminormate, normate şi multiseminormate

$(X,K,+,\cdot)$ s.l.; $p:X\to\Re$ seminormă;

Def. Semimetrică definită de seminormă

$d_p:X\times X\to\Re$, $\forall x,y\in X$, $d_p(x,y):=p(x-y)$ semimetrica definită de p;

$\tau_{dp}=\tau_p$ top. produsă de d_p; τ_{dp} compatibilă cu str. alg. a $(X,K,+,\cdot)$,

adică $(X,K,+,\cdot,\tau_p)$ s.l.t.;

Def. Spațiu seminormat

$(X,K,+,\cdot)$ s.l.; $p:X\to\Re$ seminormă $\Leftrightarrow$ (X,p) sp.sn.

Obs. (X,p) sp.sn. $\Rightarrow$ (X,K,+,·,τ_p) s.l.t.; reciproca nu e adevărată;

P.8.1 Proprietățile sm. induse de sn.

(X,K,+,·) s.l.; p:X→$\Re$ sn.; d_p sm. def. de sn.; u.a.s.e.:

(1) (X,p) s.s.Hausdorff; (2) d metrică; (3) $\forall x\in X\setminus\{O_X\}$, p(x)>0;

Dem.8.1 (1⇒2): $x\neq y \Rightarrow \exists V\in \mathcal{V}(x), U\in \mathcal{V}(y): V\cap U=\emptyset \Rightarrow (x\in V \Rightarrow x\notin U)$

$U\in \mathcal{V}(y) \Rightarrow \exists r>0 : B(y,r)\subseteq U,\ x\notin U \Rightarrow x\notin B(y,r) \Rightarrow d(x,y)\geq r>0$

(2⇒3): $x\in X\setminus\{O_X\} \Rightarrow d(x,O_X)>0 \Leftrightarrow p(x)>0$; (3⇒1): $x\neq y \Rightarrow x-y\neq O_X \Rightarrow r:=p(x-y)/2=d(x,y)/2>0$;

$B_1(x,r)\cap B_2(y,r)=\emptyset$; $B_1\in \mathcal{V}(x)$; $B_2\in \mathcal{V}(y)$;

Def. Spațiu normat

(X,K,+,·) s.l.; $\|\cdot\|$:X→$\Re$ normă $\Leftrightarrow$ (X,$\|\cdot\|$) s.n.

Def. Spațiu multiseminormat

(X,K,+,·) s.l.; p_j:X→$\Re$, $j\in J$ seminorme; fie τ_j top. generată de p_j $\Rightarrow$

$(\tau_j)_{j\in J}$ familie de top. compatibile cu "+" şi "·";

se poate defini o topologie mai fină decât τ_j, $j\in J$ comp. cu str.alg. X;

$\wp$(X) cea mai fină topologie pe X;

$B_j(x,r_0):=\{y\in X \mid p_j(y-x)<r_0\}$; $\tau_j=\{M\subseteq X \mid \forall x\in M\ \exists r>0 : B_j(x,r)\subseteq M\}$;

Fie $I\subseteq J$ ($I\in \wp(J)$); $B_I(x_0,r_0):=\cap_{i\in I}B_i(x_0,r_0)$ – multibilă;

T.8.2 Sn. indusă de o familie de sn. pe un s.l.

(X,K,+,·) s.l.; $(p_j)_{j\in J}$ fam. de sn.;

$U:X\to \wp(\wp(X))$, $U(x):=\{B_I(x,r), I\in F(J), r>0\}$, $F(J)=\{I\subseteq J, |I|<\aleph_0\}$;

u.a.s.a.:

(8.2.1) $\exists!\tau\subseteq \wp(X)$, (X,$\tau$) sp.t., $\forall x\in X \Rightarrow U(x)$ s.f.v. x în (X,τ)

(8.2.2) $\forall j\in J$, $\tau_j\subseteq\tau$

Dem.8.2 Dacă U definit astfel respectă

(U1) $\mathcal{U}(x)\neq\emptyset$; (U2) $\forall U\in \mathcal{U}(x) \Rightarrow x\in U$

(U3) $\forall U_1,U_2\subseteq \mathcal{U}(x)$, $\exists U\in \mathcal{U}(X) : U \subseteq U_1\cap U_2$

(U4) $\forall U\in \mathcal{U}(x)\ \exists U'\in \mathcal{U}(x) \mid \forall y\in U'\ \exists U''\in \mathcal{U}(y) : U''\subseteq U$

atunci $\exists!\ \tau$ pe X a.î. $\forall x\in X$, $\mathcal{U}(x)$ s.f.v. x; $\tau=\tau_U$;

(8.2.U1) $B_I(x,1)\in U(x)\neq\emptyset$;

(8.2.U2) $x\in B_i(x,r) \Rightarrow x\in\cap_{i\in I}B_i(x,r)=B_I(x,r)$;

(8.2.U3) $\forall M,N\subseteq J$, $B_M(x,r_M),B_N(x,r_N)\in U(x)$; $I:=M\cup N$;

$M,N\in F(J) \Rightarrow I=M\cup N\in F(J)$; $r:=\min\{r_N,r_M\}$;

$B_I(x,r)\subseteq B_M(x,r_M)$; $B_I(x,r)\subseteq B_N(x,r_N)$, $U:=B_I(x,r)\subseteq B_M(x,r_M)\cap B_N(x,r_N)$;

(8.2.U4) Fie $U=B_I(x,r)$; $U':=U$; Fie $y\in U' \Rightarrow p_i(y,x)<r\ \forall i\in I$;

$r_0:=\min\{r-p_i(y-x),i\in I\}=r-\max\{p_i(y-x),i\in I\}$; $r_0>0$; $U'':=B_I(y,r_0)\in \mathcal{U}(y)$;

$z\in U'' \Rightarrow \forall i\in I, p_i(z-x)=p_i(z-y+y-x)\le p_i(z-y)+p_i(y-x)<r_0+ p_i(y-x)=$

$= r+p_i(y-x)-\max\{p_i(y-x),i\in I\}\le r \Rightarrow z\in U$;

SPAȚII BANACH; B(T,K), CB(T,K), C(T,K)

Def. Normă pe s.l.

$(X,K,+,\cdot)$ s.l.; $\|\cdot\|:X\to\Re$ normă pe X $\Leftrightarrow$

(N1) $\|O_X\|=0$; $\forall x\in X\setminus\{O_X\}, \|x\|>0$;

(N2) $\forall\alpha\in K, \forall x\in X\ \|\alpha\cdot x\|=|\alpha|\cdot\|x\|$;

(N3) $\forall x,y\in X, \|x+y\|\le\|x\|+\|y\|$;

Def. Spațiu normat

$(X,K,+,\cdot)$ s.l.; $\|\cdot\|:X\to\Re$ normă pe X $\Leftrightarrow (X,\|\cdot\|)$ s.n.;

Def. Metrică generată de normă

$(X,\|\cdot\|)$ s.n.; $d_{\|\cdot\|}:X\times X\to\Re$, $d_{\|\cdot\|}(x,y):=\|x-y\|$ metrica generată de $\|\cdot\|$;

Def. Spațiu Banach

$(X,\|\cdot\|)$ s.n.; $(X,d_{\|\cdot\|})$ sp.m. complet $\Leftrightarrow (X,\|\cdot\|)$ s. Banach;

Def. Mulțimea funcțiilor mărginite B(T,K)

$B(T,K):=\{x\in F(T,K) \mid \exists M\ge0, \forall t>0 \Rightarrow x(t)\le M\}$ – m. fcț. mărginite;

T.9.1 Caracterizarea B(T,K)

$T\ne\emptyset$; u.a.s.a.:

(1) $B(T,K)\le F(T,K)$;

(2) $\|\cdot\|_\infty:B(T,K)\to\Re$, $\|x\|_\infty:=\sup\{|x(t)|, t\in T\}$ norma uniformă

(3) $(B(T,K),\|\cdot\|_\infty)$ s. Banach;

Dem.9.1:

(1): $f_0:T\to K$, $f_0(t):=O_K$; $f_0\in B(T,K)$; $x_1,x_2\in B(T,K)$; $\alpha_1,\alpha_2\in K$;

$\forall t\in T, |x(t)|=|\alpha_1x_1(t)+\alpha_2x_2(t)|\le|\alpha_1|\cdot|x_1(t)|+|\alpha_2|\cdot|x_2(t)|\le|\alpha_1|\cdot M_1+|\alpha_2|\cdot M_2$

$\Rightarrow x\in B(T,K) \Rightarrow B(T,K)\le F(T,K)$;

(2): $\|O_{B(T,K)}\|_\infty=\sup\{|O_{B(T,K)}(t)|, t\in T\}=\sup\{0, t\in T\}=0$;

$x\ne O_{B(T,K)} \Rightarrow \exists\ t_0\in T, x(t_0)\ne0$; $\|x\|_\infty\ge|x(t_0)|>0$;

$\|\alpha x\|_\infty=\sup\{|\alpha x(t)|,t\in T\}=\sup\{|\alpha|\cdot|x(t)|,t\in T\}=|\alpha|\cdot\sup\{|x(t)|,t\in T\}=$

$=|\alpha|\cdot\|x\|_\infty$; $\forall t\in T, |(x+y)(t)|=|x(t)+y(t)|\le|x(t)|+|y(t)|\le\|x\|_\infty+\|y\|_\infty\Rightarrow$(N3);

(3): Fie $(x_n)_{n\ge1}\subseteq B(T,K)$ Cauchy $\Rightarrow \forall\varepsilon>0\ \exists n_0\ge1 : \forall n,m\ge n_0$,

$\|x_n-x_m\|_\infty<\varepsilon \Rightarrow \forall\varepsilon>0\ \exists n_0\ge1 : \forall n,m\ge n_0, \forall t\in T, |x_n(t)-x_m(t)|<\varepsilon \Rightarrow$

$\forall t\in T$, $(x_n(t))_{n\geq 1}$ Cauchy în K $\Rightarrow$ $(x_n(t))_{n\geq 1}$ convergent în K $\Rightarrow$

$\forall t\in T$, $x_n(t)\to x_t$; fie $x:T\to K$, $x(t):=x_t=\lim_{n\to\infty}x_n(t)$; (?)$x\in B(T,K)$:

$\forall n\geq n_1$, $\forall t\in T$, $|x_n(t)-x_{n1}(t)|\leq\|x_n-x_{n1}\|_\infty<\varepsilon:=1$; trecem la limită în K $\Rightarrow$

$\forall t\in T$, $\lim_{n\to\infty}|x_n(t)-x_{n1}(t)|<\varepsilon=1 \Leftrightarrow \forall t\in T$, $|x(t)-x_{n1}(t)|<\varepsilon=1=M \Rightarrow$

$\exists n_1\geq 1$: $\sup\{|x(t)-x_{n1}(t)|,t\in T\}=\|x-x_{n1}\|<1 \Rightarrow x-x_{n1}\in B$; $x_{n1}\in B \Rightarrow x\in B$;

$\exists n_0$: $\forall n,m\geq n_0$, $\forall t\in T$, $|x_n(t)-x_m(t)|<\varepsilon/2$; $m\to\infty$ în K $\Rightarrow \exists n_0$: $\forall n\geq n_0$,

$\forall t\in T$, $|x(t)-x_n(t)|\leq\varepsilon/2<\varepsilon \Rightarrow \forall\varepsilon>0,\exists n_0$: $\forall n\geq n_0$, $\|x-x_n\|_\infty<\varepsilon \Rightarrow x_n\to x$;

Obs. $(B(T,K),K,+,\cdot)\leq(F(T,K),K,+,\cdot)$;

Def. Continuitate în spațiul funcțiilor peste un spațiu topologic

(T,τ) sp.t.; $T\neq\emptyset$; $t_0\in T$; $x_0\in F(T,K)$; $|\cdot|$ norma din K;

x_0 cont. $t_0 \Leftrightarrow \forall\varepsilon>0\ \exists V\in \mathcal{V}(t_0)\subseteq\tau$, $\forall t\in V\subseteq T \Rightarrow |x(t)-x_0(t)|<\varepsilon$;

Def. Mulțimea funcțiilor continue C(T,K)

(T,τ) sp.t.; $T\neq\emptyset$; $C(T,K):=\{x\in F(T,K)$, x cont. T$\}$

P.9.2 C(T,K) ≤ F(T,K)

Dem. $f_0:T\to K$, $f_0(t):=O_K$; $f_0\in C(T,K)\neq\emptyset$; $x_1,x_2\in C(T,K)$, $\alpha_1,\alpha_2\in K$,

$x:=\alpha_1x_1+\alpha_2x_2$; fie $\varepsilon>0$;

x_1, $x_2\in C(T,K)$, $t_0\in T \Rightarrow$

$\exists V_1\in \mathcal{V}(t_0)$, $\forall t\in V_1$, $|x_1(t)-x_1(t_0)|<\varepsilon/(|\alpha_1|+|\alpha_2|+1)$;

$\exists V_2\in \mathcal{V}(t_0)$, $\forall t\in V_2$, $|x_2(t)-x_2(t_0)|<\varepsilon/(|\alpha_1|+|\alpha_2|+1)$;

$V:=V_1\cap V_2 \Rightarrow V\in \mathcal{V}(t_0)$; $\forall t\in V$, $|x(t)-x(t_0)|=$

$=|\alpha_1x_1(t)+\alpha_2x_2(t)-\alpha_1x_1(t_0)-\alpha_2x_2(t_0)|\leq(|\alpha_1|+|\alpha_2|)\varepsilon/(|\alpha_1|+|\alpha_2|+1)<\varepsilon$

$\Rightarrow x\in C(T,K) \Rightarrow C(T,K)\leq F(T,K)$;

Def. Mulțimea funcțiilor continue mărginite CB(T,K)

(T,τ) sp.t.; $T\neq\emptyset$; $CB(T,K):=C(T,K)\cap B(T,K)$;

Obs. CB(T,K)≤F(T,K)

$C(T,K)\leq F(T,K)$, $B(T,K)\leq F(T,K)$,

$C(T,K)\cap B(T,K) \Rightarrow CB(T,K)\leq F(T,K)$

Def. Continuitate între două spații metrice

(X,d_X), (Y,d_Y) sp.m.; $f:X\to Y$;

$$f\in C(X,Y) \Leftrightarrow \forall x\in X,\ \forall\varepsilon>0,\ \exists\delta>0,\ f(B(x,\delta))\subseteq B(f(x),\varepsilon)$$

Def. Uniform continuitate între două spații metrice

(X,d_X), (Y,d_Y) sp.m.; $f:X\to Y$;

$$f \text{ uniform continuă} \Leftrightarrow \forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in X,\ f(B(x,\delta))\subseteq B(f(x),\varepsilon)$$

T.9.3 Weierstrass

(X,d_X), (Y,d_Y) sp.m.; $f\in C(X,Y)$; (X,d_X) compact $\Rightarrow$ f unif. continuă;

T.9.4 CB(T,K) s.s.B. B(T,K)

Dem.4.3 $(B(T,K),K,+,\cdot,\tau_U)$; (?) $CB(T,K)\in\underline{\tau}_U$; Fie $x\in cl_{B(T,K)}CB(T,K)$

$\Rightarrow x\in B(T,K)$; Fie $\varepsilon>0, B_{\|\cdot\|\infty}(x,{}^{\varepsilon}/_3)=\{y\in B(T,K), \|y-x\|_\infty<{}^{\varepsilon}/_3\}\in \mathcal{V}_{B(T,K)}(x)$

$clB=\{x\in X|\forall V\in \mathcal{V}(x)\Rightarrow V\cap B\neq\emptyset\}\Rightarrow CB(T,K)\cap B_{\|\cdot\|\infty}(x,{}^{\varepsilon}/_3)\neq\emptyset$;

fie $y\in CB(T,K)\cap B_{\|\cdot\|\infty}(x,{}^{\varepsilon}/_3)$; fie $t_0\in T\Rightarrow$ y cont. $t_0\Rightarrow$

$\Rightarrow \exists V\in \mathcal{V}(t_0), \forall t\in V, |y(t)-y(t_0)|<{}^{\varepsilon}/_3$;

$\forall t\in T, |x(t)-x(t_0)|=|x(t)-y(t)+y(t)-y(t_0)+y(t_0)-x(t_0)|\leq|x(t)-y(t)|+$

$+|y(t)-y(t_0)|+|y(t_0)-x(t_0)|\leq\|x-y\|_\infty+{}^{\varepsilon}/_3+\|y-x\|_\infty<{}^{\varepsilon}/_3+{}^{\varepsilon}/_3+{}^{\varepsilon}/_3=\varepsilon\Rightarrow$ x cont. t_0

$\Rightarrow x\in C(T,K)\Rightarrow x\in CB(T,K)\Rightarrow cl_{B(T,K)}CB(T,K)\subseteq CB(T,K)$;

C.9.5 (T,τ) sp.t.; $T\neq\emptyset$; T compact $\Rightarrow (C(T,K),\|\cdot\|_\infty)$ s.Banach;

Dem. T compact, T.4.3 $\Rightarrow CB(T,K)=C(T,K)$; arătăm că orice funcție continuă pe compact este și mărginită:

$K=\Re$ evident din T. Weierstrass; K=C:

$\forall t_1,t_2\in T, |re(x(t_1))-re(x(t_2))|=|re(x(t_1)-x(t_2))|\leq|x(t_1)-x(t_2)|$

$\forall t_1,t_2\in T, |im(x(t_1))-im(x(t_2))|=|im(x(t_1)-x(t_2))|\leq|x(t_1)-x(t_2)|$

$re,im:T\to\Re$ cont.; T.Weierstrass $\Rightarrow$

$\exists M_1,M_2\in\Re_+, \forall t\in T, re(x(t))<M_1, im(x(t))<M_2$;

$\forall t\in T, |x(t)|=\sqrt{(re^2(x(t))+im^2(x(t)))}<\sqrt{(M_1^2+M_2^2)}:=M\Rightarrow x\in B(T,K)$;

Spațiile Banach $(l_\infty,K,+,\cdot)$, $(c,K,+,\cdot)$, $(c_0,K,+,\cdot)$, $(l_p,K,+,\cdot)$, $(l_0,K,+,\cdot)$

$l_0:=\{(x_i)_{i\geq1}\in(K), |\{i\geq1, x_i\neq0\}|<\aleph_0\}$ șiruri nule aproape peste tot;

$l_p:=\{(x_i)_{i\geq1}\in(K), \Sigma|x_i|^p<\infty\}$, $0<p<1$ – șiruri cu seria puterilor *p* conv.;

$l_\infty:=\{(x_i)_{i\geq1}\in(K), \exists M\geq0, \forall i\geq1, |x_i|\leq M\}$ – șiruri mărginite;

$c:=\{(x_i)_{i\geq1}\in(K), \exists x\in K, x_i\to x\}$ – șiruri convergente;

$c_0:=\{(x_i)_{i\geq1}\in(K), x_i\to0\}$ – șiruri convergente la 0;

$s:=\{(x_i)_{i\geq1}\in(K)\}$ – șiruri; s=(K);

$\|\cdot\|_\infty:(K)\to\Re, \|x\|_\infty:=\sup\{|x_i|, i\geq1\}$ – normă (norma uniformă pe (K))

T.9.6 Relații între spațiile c_0, c, l_∞ și (K)

(K) s.n.; $(c_0,\|\cdot\|_\infty)\leq(c,\|\cdot\|_\infty)\leq(l_\infty,\|\cdot\|_\infty)\leq((K),\|\cdot\|_\infty)$; c_0,c,l_∞ - s. Banach;

P.9.7 Relații între spațiile l_p, l_0 și c_0

$(l_p,\|\cdot\|_\infty)\leq(c_0,\|\cdot\|_\infty)$, $(l_0,\|\cdot\|_\infty)\leq(c_0,\|\cdot\|_\infty)$; $(l_0,\|\cdot\|_\infty),(l_p,\|\cdot\|_\infty)$ - $\neg$ s. Banach;

Dem.9.6: "$\leq$" evident; s.s.B.:

(T.4.1,T:=N^*) $\Rightarrow (l_\infty,K,+,\cdot,\|\cdot\|_\infty)\leq((K),K,+,\cdot,\|\cdot\|_\infty)$;

orice şir convergent e mărginit $\Rightarrow c\subseteq l_\infty \Rightarrow c\leq l_\infty$; $c_0\subseteq c \Rightarrow c_0\leq c$;

(?) $cl_{l\infty}c=c$ ($\Leftrightarrow$ c s.s.B. l_∞); $x\in cl_{l\infty}c;\varepsilon>0;B_{\|\cdot\|\infty}(x,{}^{\varepsilon}/_{3})\in V_{l\infty}(x)\Rightarrow c\cap B\neq\emptyset$

Fie $y=(y_i)_{i\geq1}\in c\cap B \Rightarrow (y_i)$ conv. $\Rightarrow (y_i)$ Cauchy $\Rightarrow \exists n_0$: $\forall n,m\geq n_0$,

$|y_n-y_m|<{}^{\varepsilon}/_{3}$; $\forall m,n\geq n_0$, $|x_n-x_m|=|x_n-y_n+y_n-y_m+y_m-x_m|\leq|x_n-y_n|+|y_n-y_m|+$

$+|y_m-x_m|<\|x-y\|_\infty+{}^{\varepsilon}/_{3}+\|x-y\|_\infty<\varepsilon \Rightarrow$

$(x_i)_{i\geq1}$ Cauchy; X=(K) $\Rightarrow (x_i)_{i\geq1}$ conv. $\Rightarrow x\in c \Rightarrow cl_{l\infty}c\subseteq c$;

(?) $cl_cc_0=c_0$ ($\Leftrightarrow c_0$ s.s.B.c); $x\in cl_cc_0;\varepsilon>0;B_{\|\cdot\|\infty}(x,{}^{\varepsilon}/_{2})\in V_c(x)\Rightarrow c_0\cap B\neq\emptyset$

Fie $y=(y_i)_{i\geq1}\in c_0\cap B \Rightarrow y_n\rightarrow0 \Rightarrow \exists n_0$: $\forall n,m\geq n_0$, $|y_n|<{}^{\varepsilon}/_{2}$;

$\forall n\geq n_0$, $|x_n|\leq|x_n-y_n|+|y_n|<\|x-y\|_\infty+{}^{\varepsilon}/_{2}<\varepsilon \Rightarrow x_n\rightarrow0 \Rightarrow x\in c_0 \Rightarrow cl_cc_0\subseteq c_0$;

Dem.9.7: "$\leq$" evident; s.s.B.:

(?) $x_n\in l_p$; x_n Cauchy; $\lim_{n\to\infty}x_n\notin l_p$; (Cauchy, $\neg$conv. $\Rightarrow$ $\neg$complet)

$a=(a_k)_{k\geq1}$; $a_k:=k^{-1/p}$; $x_n:=(a_1,\ldots,a_n,0,\ldots)$;

$\lim_{k\to\infty}x_{n,k}=\lim_{k\to\infty}0=0 \Rightarrow x_n\in c_0$ *(1)*

$\forall n\in N^*$, $\Sigma_{i\geq1}|x_{n,i}|^p=\Sigma_{1\leq k\leq n}n^{-1}\leq\Sigma_{n\geq1}1=n(n+1)/2<\infty \Rightarrow \forall n\geq1$, $x_n\in l_p$ *(2)*

$\varepsilon>0$; $\forall n\geq n_0:=[1/\varepsilon^p]+1$ ($n_0^{-1/p}<\varepsilon$), $\forall m>0$,

$x_{n+m}-x_n=(0,\ldots0,(n+1)^{-1/p},\ldots,(n+m)^{-1/p},0,\ldots)$;

$\|x_{n+m}-x_n\|_\infty=\sup\{(n+i)^{-1/p},1\leq i\leq m\}=(n+1)^{-1/p}<n^{-1/p}\leq n_0^{-1/p}<\varepsilon \Rightarrow$

$\Rightarrow (x_n)_{n\geq1}$ Cauchy *(3)*

$\lim_{n\to\infty}x_n=a=(a_k)_{k\geq1}=(k^{-1/p})_{k\geq1}$; $\lim_{k\to\infty}a_k=\lim_{k\to\infty}k^{-1/p}=0 \Rightarrow a\in c_0$ *(4)*

$a=(a_k)_{k\geq1}$; $a_k=k^{-1/p}$; $\Sigma_{k\geq1}|a_k|^p=\Sigma_{k\geq1}k^{-1}=\infty \Rightarrow a\notin l_p$ *(5)*

Din *(1-5)* avem: $x_n\in c_0$; $x_n\in l_p$; (x_n) Cauchy; $\lim_{n\to\infty}x_n\in c_0$;

$\lim_{n\to\infty}x_n\notin l_p \Rightarrow l_p$ nu e completă în $c_0 \Rightarrow (l_p,K,+,\cdot,\|\cdot\|\infty)$ $\neg$ s. Banach;

(?) $x_n\in l_0$; x_n Cauchy; $\lim_{n\to\infty}x_n\notin l_0$; (Cauchy, $\neg$conv. $\Rightarrow$ $\neg$complet)

$b=(b_k)_{k\geq1}$; $b_k:=k^{-1}$; $y_n=(b_1,\ldots,b_n,0,\ldots)$;

$\lim_{k\to\infty}y_{n,k}=\lim_{k\to\infty}0=0 \Rightarrow y_n\in c_0$ *(6)*

$y_n=(1^{-1},2^{-1},\ldots,n^{-1},0,\ldots)$; $\forall n\geq1$, $|\{i\geq1, y_i\neq0\}|=n<\aleph_0 \Rightarrow y_n\in l_0$ *(7)*

$\varepsilon>0$; $\forall n\geq n_0:=[1/\varepsilon]+1$ ($n_0^{-1}<\varepsilon$), $\forall m>0$,

$y_{n+m}-y_n=(0,\ldots0,(n+1)^{-1},\ldots,(n+m)^{-1},0,\ldots)$;

$\|y_{n+m}-y_n\|_\infty=\sup\{(n+i)^{-1},1\leq i\leq m\}=(n+1)^{-1}<n^{-1}\leq n_0^{-1}<\varepsilon\Rightarrow(y_n)$ Cauchy *(8)*

$\lim_{n\to\infty}y_n=b=(b_k)_{k\geq1}=(k^{-1})_{k\geq1}$; $\lim_{k\to\infty}b_k=\lim_{k\to\infty}k^{-1}=0 \Rightarrow b\in c_0$ *(9)*

$\lim_{n\to\infty}y_n=(b_k)_{k\geq1}=(k^{-1})_{k\geq1}$; $|\{i\geq1, i^{-1}\neq0\}|=\aleph_0 \Rightarrow b\notin l_0$ *(10)*

Din *(6-10)* avem: $y_n \in c_0$; $y_n \in l_0$; (y_n) Cauchy; $\lim_{n\to\infty} y_n \in c_0$;

$\lim_{n\to\infty} x_n \notin l_0 \Rightarrow l_0$ nu e completă în $c_0 \Rightarrow (l_0,K,+,\cdot,\|\cdot\|\infty)$ ⌉s. Banach;

T.10.1 Inegalitatea Young şi inegalitatea Holder

Fie $p,q>1$, $p^{-1}+q^{-1}=1$; u.a.s.a.:

(Young) $a,b\geq 0 \Rightarrow ab\leq p^{-1}a^p+q^{-1}b^q$;

(Hölder) $(a_i)_{1\leq i\leq m},(b_i)_{1\leq i\leq m}\geq 0 \Rightarrow \Sigma_{1\leq i\leq m}a_ib_i\leq(\Sigma_{1\leq i\leq m}a_i^p)^{1/p}(\Sigma_{1\leq i\leq m}b_i^q)^{1/q}$;

Dem.Young; $p^{-1}+q^{-1}=1 \Rightarrow p^{-1}=1-q^{-1}$; $p=(1-q^{-1})^{-1}=q/(q-1)$;

$f(x):=ax-p^{-1}a^p-q^{-1}x^q$; ${}^{\partial f}/_{\partial x}(x)=a-q\cdot q^{-1}\cdot x^{q-1}=a-x^{q-1}=0 \Rightarrow x_v=a^{1/(q-1)}$;

${}^{\partial f(x)}/_{\partial x}(0)=a>0 \Rightarrow x_v=a^{1/(q-1)}=x_{max}$; $f(x_{max})=f_{max}=f(a^{1/(q-1)}) \Rightarrow$

$f(x)\leq f(a^{1/(q-1)})=a^{q/(q-1)}-p^{-1}a^p-q^{-1}a^{q/(q-1)}=a^{q/(q-1)}(1-q^{-1})-a^pp^{-1}=0 \Rightarrow f(b)\leq 0$;

Dem.Hölder; $A:=(\Sigma a_i^p)^{1/p}$; $B:=(\Sigma b_i^q)^{1/q}$;

$A=0$ sau $B=0$ evident adevărat; $A\neq 0$ şi $B\neq 0 \Rightarrow$ Young$(a_i/A,b_i/B) \Rightarrow$

$a_ib_i(AB)^{-1}\leq p^{-1}a_i^pA^{-p}+q^{-1}b_i^qB^{-q} \Rightarrow (AB)^{-1}\Sigma a_ib_i\leq p^{-1}A^{-p}\Sigma a_i^p+q^{-1}B^{-q}\Sigma b_i^q$

$\Rightarrow(AB)^{-1}\Sigma a_ib_i\leq p^{-1}(\Sigma a_i^p)^{-1}\Sigma a_i^p+q^{-1}(\Sigma b_i^q)^{-1}\Sigma b_i^q=p^{-1}+q^{-1}=1 \Rightarrow \Sigma a_ib_i\leq AB$

T.10.2 Inegalitatea Minkovski

$p\geq 1$, $(a_i)_{1\leq i\leq m},(b_i)_{1\leq i\leq m}\geq 0 \Rightarrow (\Sigma(a_i+b_i)^p)^{1/p}\leq(\Sigma a_i^p)^{1/p}+(\Sigma b_i^p)^{1/p}$;

Dem.Minkovski: $p=1 \Rightarrow \Sigma a_i+b_i=\Sigma a_i+\Sigma b_i$ adev.; $p>1 \Rightarrow q:=(1-p^{-1})^{-1}$;

$B:= (\Sigma a_i^p)^{1/p}+(\Sigma b_i^p)^{1/p}$; $A:=\Sigma(a_i+b_i)^p$; $A=0\leq(\Sigma a_i^p)^{1/p}+(\Sigma b_i^p)^{1/p}$ adev.; $A\neq 0 \Rightarrow A=\Sigma(a_i+b_i)(a_i+b_i)^{p-1}=\Sigma a_i(a_i+b_i)^{p-1}+\Sigma b_i(a_i+b_i)^{p-1}\leq_{Hölder}$

$\leq(\Sigma a_i^p)^{1/p}(\Sigma(a_i+b_i)^{(p-1)q})^{1/q}+(\Sigma b_i^p)^{1/p}(\Sigma(a_i+b_i)^{(p-1)q})^{1/q}=(\Sigma(a_i+b_i)^p)^{1/q}B$

$\Rightarrow A\leq A^{1/q}B \Rightarrow A^{1-1/q}\leq B \Rightarrow A^{1/p}\leq B$: Minkovski;

C.10.3 Norma Minkovski peste K^m

$p\geq 1$; $\|\cdot\|_p:K^m\to\Re$, $\|x\|_p:=(\Sigma|x_i|^p)^{1/p}$, normă

Dem.

(N1): $\|x\|_p=(\Sigma|x_i|^p)^{1/p}=0 \Leftrightarrow x=(x_1,\ldots,x_m)=(0,\ldots,0)$;

(N2): $\alpha\in K$, $x\in K^m$, $\|\alpha x\|_p=(\Sigma|\alpha x_i|^p)^{1/p}=|\alpha|(\Sigma|x_i|^p)^{1/p}=|\alpha|\cdot\|x\|_p$;

(N3): $\forall i$: $|x_i+y_i|^p\leq(|x_i|+|y_i|)^p \Rightarrow (\|x+y\|_p)^p=\Sigma|x_i+y_i|^p\leq\Sigma(|x_i|+|y_i|)^p \Rightarrow_M$

$\|x+y\|_p=(\Sigma|x_i+y_i|^p)^{1/p}\leq(\Sigma(|x_i|+|y_i|)^p)^{1/p}\leq(\Sigma|x_i|^p)^{1/p}+(\Sigma|y_i|^p)^{1/p}=\|x\|_p+\|y\|_p$

P.10.4 Norma Minkovski peste s.B. l_p

$p\geq 1$; $\|\cdot\|_p:l_p\to\Re$, $\|x\|_p:=(\Sigma|x_i|^p)^{1/p}$, normă

Dem.

(N1): $\|x\|_p=(\Sigma|x_i|^p)^{1/p}=0 \Leftrightarrow x=(x_1,\ldots,x_m)=(0,\ldots,0)$;

(N2): $\alpha\in K$, $x\in l_p$, $\|\alpha x\|_p=(\Sigma|\alpha x_i|^p)^{1/p}=|\alpha|(\Sigma|x_i|^p)^{1/p}=|\alpha|\cdot\|x\|_p$;

(N3): $\forall i$: $|x_i+y_i|^p \le (|x_i|+|y_i|)^p \Rightarrow (\|x+y\|_p)^p = \Sigma|x_i+y_i|^p \le \Sigma(|x_i|+|y_i|)^p \Rightarrow_M$

$\Rightarrow \forall m \ge 1$, $\Sigma_{1\le i\le m}|x_i+y_i|^p \le (\Sigma_{1\le i\le m}|x_i|^p)^{1/p} + (\Sigma_{1\le i\le m}|y_i|^p)^{1/p}$; $m\to\infty \Rightarrow$ (N3)